Scaling Algebras and Renormalization Group in Algebraic Quantum Field Theory

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Abstract

For any given algebra of local observables in Minkowski space an associated scaling algebra is constructed on which renormalization group (scaling) transformations act in a canonical manner. The method can be carried over to arbitrary spacetime manifolds and provides a framework for the systematic analysis of the short distance properties of local quantum field theories. It is shown that every theory has a (possibly non-unique) scaling limit which can be classified according to its classical or quantum nature. Dilation invariant theories are stable under the action of the renormalization group. Within this framework the problem of wedge (Bisognano-Wichmann) duality in the scaling limit is discussed and some of its physical implications are outlined.

1 Introduction

The algebraic approach to relativistic quantum field theory has proven to be an efficient setting for the structural analysis of properties of physical systems which manifest themselves at the upper end of the spatio-temporal scale. Examples are the classification of the possible statistics and superselection structure of particles, collision theory and the clarification of the infrared properties of theories with long range forces [1]. At the lower end of the scale the algebraic point of view has been less successful, however. As a matter of fact, basic ideas which have emerged from physics at small scales such as the parton picture or the notion of asymptotic freedom have not yet found an appropriate expression in the algebraic setting. What is missing so far in this approach is the analogue of the renormalization group, cf. [2] and references quoted there, which allows one to transform a theory at given scale into the corresponding theories at other scales.

In order to understand the origin of this difficulty one has to call to mind the conceptual foundations of the algebraic approach. Algebraic quantum field

theory is based on the idea that the correspondence

$$\mathcal{O} \to \mathfrak{A}(\mathcal{O}) \tag{1.1}$$

between spacetime regions $\mathcal{O} \subset \mathbb{R}^4$ and local algebras of observables $\mathfrak{A}(\mathcal{O})$ constitute the intrinsic description of a theory [1]. One therefore refrains from assigning to the individual elements of these algebras any particular physical interpretation. In fact, one presumes that this interpretation is essentially fixed once the map is given. We adhere in the following to the standard terminology according to which this map is called a local net.

Quantum fields, which are a basic ingredient in the conventional approach to the renormalization group, are regarded as a kind of coordinatization of the local algebras and therefore do not appear explicitly in the algebraic setting. This view is justified by the observation that different irreducible sets of field operators which are relatively local to each other yield the same scattering matrix [3]. Thus the physical content of a theory does not depend on a particular choice of fields. All what matters is the information about the localization properties of the operators, i.e., the net.

The absence of quantum fields in the algebraic setting causes problems, however, if one wants to apply the ideas of the renormalization group. In the conventional framework of quantum field theory the renormalization group transformations $R_\lambda, \lambda > 0$, act on the underlying quantum fields $\phi(x)$ by scaling the spacetime coordinates x, accompanied by a multiplicative renormalization, $R_\lambda : \phi(x) \to \phi_\lambda(x) \doteq N_\lambda \, \phi(\lambda x)$. One thereby maps the theory at the original scale $\lambda = 1$, say, onto the corresponding theory at scale λ whithout changing the value of the fundamental physical constants, i.e., the velocity of light c and Planck's constant $\hbar$. Moreover, by the multiplicative renormalization factor N_λ, the scale of field strength is adjusted in such a way that the mean values and mean square fluctuations of the fields in some fixed reference state are of the same order of magnitude at small scales. Thus the quantum fields are employed to identify at each scale λ a set of operators with a fixed physical interpretation. These operators can then be used to compare the properties of the theory at different scales.

It is apparent that this approach is at variance with the basic philosophy of algebraic quantum field theory and one is faced with the question of how to implement the renormalization group in this setting. It is the aim of the present article to provide a solution of this conceptual problem and to establish a mathematical framework apt for the structural analysis of local nets at small scales. Our approach is based on the following elementary observations.

(i) According to the geometrical significance of the renormalization group the transformations R_λ should map the given net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ at the original spatio-temporal scale 1 onto the corresponding net $\mathcal{O} \to \mathfrak{A}_\lambda(\mathcal{O}) \doteq \mathfrak{A}(\lambda\mathcal{O})$ at scale λ, i.e.,

$$R_\lambda : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}_\lambda(\mathcal{O}) \tag{1.2}$$

for every region $\mathcal{O} \subset \mathbb{R}^4$. Since space and time are scaled in the same way the value of the velocity of light c is kept fixed under these maps.

(ii) The condition that $\hbar$ remains constant under renormalization group transformations can be expressed in the algebraic setting as follows. If one scales space and time by λ and does not want to change the unit of action one has to rescale energy and momentum by λ^{-1}. The energy-momentum scale can be set by determining the energy and momentum which is transferred by the action of observables to physical states. Hence if [1] $\widetilde{\mathfrak{A}}(\widetilde{\mathcal{O}})$ denotes the subspace of all (quasi local) observables which, at the original scale 1, can transfer energy-momentum contained in the set $\widetilde{\mathcal{O}} \subset \mathbb{R}^4$ and if $\widetilde{\mathfrak{A}}_\lambda(\widetilde{\mathcal{O}}) \doteq \widetilde{\mathfrak{A}}(\lambda^{-1}\widetilde{\mathcal{O}})$ denotes the corresponding space at scale λ, then the transformations R_λ should induce a map

$$R_\lambda : \widetilde{\mathfrak{A}}(\widetilde{\mathcal{O}}) \to \widetilde{\mathfrak{A}}_\lambda(\widetilde{\mathcal{O}}) \tag{1.3}$$

for every $\widetilde{\mathcal{O}}$. (An analogous relation should hold for the angular momentum transfer.)

(iii) In the case of dilation invariant theories the transformations R_λ are expected to be isomorphisms, yet this will not be true in general since the algebraic relations between observables may depend on the scale. But since the transformations R_λ are designed to identify observables at different scales they still ought to be continuous, bounded maps, uniformly in λ. This condition is akin to the multiplicative renormalization of fields in the conventional approach to the renormalization group.

The above conditions subsume the physical constraints imposed on the renormalization group transformations R_λ, although they do not fix these maps. As a matter of fact, there exists an abundance of such maps for any given $\lambda > 0$. But all of these maps identify the same net at scale λ, they merely reshuffle the operators within the local algebras in different ways. Bearing in mind the basic hypothesis of algebraic quantum field theory according to which the physical information of a theory is contained in the net, it should thus not matter which map one picks for the short distance analysis of a theory. One may consider any one of them or, what amounts to the same thing, one may consider them all.

We adopt here the latter point of view which can conveniently be expressed by introducing the concept of scaling algebra. Roughly speaking, the scaling algebra consists of operator valued functions $\lambda \to R_\lambda(A)$, $\lambda > 0$, which are the orbits of the local observables A under the action of all admissible transformations R_λ. As we shall see, the specific properties of the transformations R_λ indicated above imply that the scaling algebra still has the structure of a local net on which the Poincaré group acts in a continuous manner. Moreover, the renormalization group induces an additional symmetry of this net: scaling transformations. Thus the physically significant features of the renormalization group give rise to specific algebraic properties of the scaling algebra.

The states of physical interest can be lifted to the scaling algebra and their behaviour under scaling transformations can then be analyzed. We will show that

[1] All quantities relating to four-momentum space $\mathbb{R}^4$ will be marked by a tilde $\widetilde{\ }$, cf. the subsequent section for precise definitions.

the transformed states have, at arbitrarily small scales, limits which are vacuum states. This result allows it on one hand to classify the possible scaling limits of local nets. Besides the cases corresponding to theories with an ultraviolet fixed point there appear in the present general setting other possibilities, such as theories with a classical or non-unique scaling limit. If the underlying theory is invariant under dilations and satisfies a compactness criterion proposed by Haag and Swieca [4], it is invariant under the action of the renormalization group and coincides with its scaling limit.

The information that the scaling limits of physical states are always vacuum states provides, on the other hand, the basis for a more detailed short distance analysis. A relevant technical result in this respect is the observation that theories satisfying a condition of wedge duality, established by Bisognano and Wichmann [5], comply with this condition also in the scaling limit. In an application of this result we will generalize a theorem by Fredenhagen [6] and show that in theories with a non-classical scaling limit the local von Neumann algebras $\mathfrak{A}(\mathcal{O})^-$ corresponding to double cones $\mathcal{O}$ are of type III_1 according to the classification of Connes. This fact has interesting physical implications, cf. for example [7, 8].

The condition of wedge duality is also of vital importance in the general analysis of the superselection structure, carried out by Doplicher, Haag and Roberts [1, 9]. Applying these methods in the present setting one can determine the gauge group and the particle structure appearing in the scaling limit and thereby lay the ground for a rigorous discussion of the parton aspects of the theory. We will deal with this issue in a future publication.

The present article is organized as follows. In the subsequent Sec. 2 our assumptions are stated and some preliminary results on the momentum space properties of local observables are derived. The scaling algebra and the accompanying scaling transformations are introduced in Sec. 3, where it is also shown how to lift physical states to this algebra and to recover from the lifts the physical information. This method can be generalized to nets on arbitrary spacetime manifolds, as is outlined in the Appendix. Section 4 contains the analysis of the scaling limits of physical states and the resulting classification of theories. The special case of dilation invariant theories is discussed in Sec. 5, and Sec. 6 contains the derivation of wedge duality in the scaling limit, as well as a discussion of its consequences. The article concludes with an outlook on further developments of the theory.

2 The structure of local observables

For the convenience of the reader who is not familiar with the framework of algebraic quantum field theory [1] we briefly list in the first part of this section the basic assumptions and add a few comments. In the second part we establish some properties of local operators in momentum space which are of relevance in the subsequent discussion.

1. *(Locality)* We suppose that the local observables of the underlying theory

generate a local net over Minkowski space $\mathbb{R}^4$, that is an inclusion preserving map $\mathcal{O} \rightarrow \mathfrak{A}(\mathcal{O})$ from the set of open, bounded regions $\mathcal{O} \subset \mathbb{R}^4$ to unital C*-algebras $\mathfrak{A}(\mathcal{O})$ on the pertinent Hilbert space $\mathcal{H}$. Thus each $\mathfrak{A}(\mathcal{O})$ is a norm closed subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ which is stable under taking adjoints and contains the unit operator, and there holds

$$\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2. \tag{2.1}$$

The net is supposed to comply with the principle of locality (Einstein causality) according to which observables in spacelike separated regions commute. In formula form,

$$\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)' \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2', \tag{2.2}$$

where $\mathcal{O}'$ denotes the spacelike complement of $\mathcal{O}$ and $\mathfrak{A}(\mathcal{O})'$ the set of operators in $\mathcal{B}(\mathcal{H})$ which commute with all operators in $\mathfrak{A}(\mathcal{O})$. The (global) algebra generated by all local algebras $\mathfrak{A}(\mathcal{O})$ (as a norm inductive limit) will be denoted by $\mathfrak{A}$ and is assumed to act irreducibly on $\mathcal{H}$.

2. *(Covariance)* On $\mathcal{H}$ there exists a continuous unitary representation U of the Poincaré group $\mathcal{P}_+^\uparrow$ which induces automorphisms of the net. Thus for each $(\Lambda, x) \in \mathcal{P}_+^\uparrow$ there is an $\alpha_{\Lambda,x} \in \mathrm{Aut}(\mathfrak{A})$ given by

$$\alpha_{\Lambda,x}(A) \doteq U(\Lambda, x) A U(\Lambda, x)^{-1}, \quad A \in \mathfrak{A}, \tag{2.3}$$

and, in an obvious notation,

$$\alpha_{\Lambda,x}(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\Lambda\mathcal{O} + x) \tag{2.4}$$

for any region $\mathcal{O}$. We amend this fundamental postulate by the condition that the operator valued functions

$$(\Lambda, x) \rightarrow \alpha_{\Lambda,x}(A), \ A \in \mathfrak{A} \tag{2.5}$$

are continuous in the norm topology. This assumption, which plays a crucial role in the present investigation, does not impose any essential restrictions of generality. For these functions are always continuous in the strong operator topology, because of the continuity properties of the representation U and the fact that the operators A are bounded. Hence, by convolution of these operator valued functions with suitable test functions, one can always proceed to a local net which complies with our continuity condition and which is dense in the original net in the strong operator topology. So it still contains the relevant information about the states of interest here.

For the sake of uniqueness we also assume that the local algebras $\mathfrak{A}(\mathcal{O})$ are maximal in the following sense: any operator in the strong operator topology closure $\mathfrak{A}(\mathcal{O})^-$ of $\mathfrak{A}(\mathcal{O})$ which complies with our continuity condition is already contained in $\mathfrak{A}(\mathcal{O})$. This condition on the net can always be satisfied by enlarging the local algebras, if necessary.

3. *(Spectrum condition)* The joint spectrum of the generators of the unitary representation of the translations $U\,|\,\mathbb{R}^4$ is contained in the closed forward lightcone $\overline{V}_+$. Moreover, there is an (up to a phase unique) unit vector $\Omega \in \mathcal{H}$, representing the vacuum, which is invariant under the action of the representation U,

$$U(\Lambda, x)\Omega = \Omega, \quad (\Lambda, x) \in \mathcal{P}_+^\uparrow. \tag{2.6}$$

Besides the vacuum Ω and its local excitations, described by the vectors in $\mathcal{H}$, there exist other states of physical interest such as charged states, thermal states, etc. These states are not represented by vectors in $\mathcal{H}$, but can be described by suitable positive, linear and normalized functionals ω on the given algebra $\mathfrak{A}$. By the GNS-construction [1], any such functional gives rise to a representation π_ω of $\mathfrak{A}$ on some Hilbert space $\mathcal{H}_\omega$, and there exists a cyclic vector[2] $\Omega_\omega \in \mathcal{H}_\omega$ such that

$$\omega(A) \;=\; (\Omega_\omega, \pi_\omega(A)\Omega_\omega),\;\; A \in \mathfrak{A}. \tag{2.7}$$

We note that according to this view the vector states in the original Hilbert space $\mathcal{H}$ induce the identical representation of $\mathfrak{A}$. Since this representation is distinguished by the existence of a vacuum state, we call it vacuum representation. It may be regarded as the defining (faithful) representation of the theory, in accord with common practice in the actual construction of field theoretic models based on vacuum functionals.

The states of physical interest correspond to functionals ω on $\mathfrak{A}$ which are locally normal with respect to the vacuum representation [1, Sec. III.3.1]. We recall that this property implies, in view of the Reeh-Schlieder-Theorem, that the restrictions of these functionals to any local algebra, $\omega|\,\mathfrak{A}(\mathcal{O})$, can be represented by vectors $\Omega_\mathcal{O}$ in the vacuum Hilbert space $\mathcal{H}$ [1, Sec. V.2.2],

$$\omega(A) = (\Omega_\mathcal{O}, A\Omega_\mathcal{O}), \quad A \in \mathfrak{A}(\mathcal{O}). \tag{2.8}$$

This fact is of interest in the present investigation since it shows that for the short distance analysis of physical states it suffices to consider the vector states in the vacuum representation.

We turn now to the analysis of the momentum space properties of the operators $A \in \mathfrak{A}$. To this end we consider the Fourier transforms of the operator valued functions $x \to \alpha_x(A)$, $x \in \mathbb{R}^4$, which are defined in the sense of distributions. In fact, for each $f \in L^1(\mathbb{R}^4)$ the expression

$$\alpha_f(A) \doteq \int dx\, f(x)\alpha_x(A) \tag{2.9}$$

exists as a Bochner integral in $\mathfrak{A}$ because of the norm continuity of $x \to \alpha_x(A)$, and $||\alpha_f(A)|| \leq ||f||_1||A||$.

Definition: Let $A \in \mathfrak{A}$. The support of A in momentum space is the smallest closed subset $\widetilde{\mathcal{O}} \subset \mathbb{R}^4$ such that $\alpha_f(A) = 0$ for all $f \in L^1(\mathbb{R}^4)$ with supp$\widetilde{f} \subset$

[2] This means that $\pi_\omega(\mathfrak{A})\Omega_\omega$ is dense in $\mathcal{H}_\omega$

$\mathbb{R}^4 \backslash \widetilde{\mathcal{O}}$, where $\widetilde{f}$ denotes the Fourier transform of f. (In more physical terms, $\widetilde{\mathcal{O}}$ may be called the energy-momentum transfer of A.) The subspace of all operators $A \in \mathfrak{A}$ with support in momentum space in a given set $\widetilde{\mathcal{O}} \subset \mathbb{R}^4$ is denoted by $\widetilde{\mathfrak{A}}(\widetilde{\mathcal{O}})$.

The condition of locality imposes strong restrictions on the support properties of local operators A in momentum space. Let us first remark that the support cannot be a compact set, because otherwise the functions $x \rightarrow \alpha_x(A)$ would be entire analytic. This in turn would imply that the commutator functions $x \rightarrow [\alpha_x(A), B]$, being 0 for any local operator $B \in \mathfrak{A}$ and sufficiently large spacelike x, would vanish for all x. In view of the irreducibility of $\mathfrak{A}$, this is only possible if A is a multiple of the identity. By a similar argument it also follows that the support cannot be contained in any salient cone of $\mathbb{R}^4$.

It does not seem simple to specify in more explicit terms the possible support properties of (bounded) local operators in momentum space. But we need no such detailed information. What is of interest here is the fact that there exist in each local algebra $\mathfrak{A}(\mathcal{O})$ operators whose support in momentum space is all of $\mathbb{R}^4$. As a matter of fact this case is generic in a certain sense, as will become clear from the subsequent argument.

Lemma 2.1 *Suppose that the local algebras are non-trivial for every open spacetime region, $\mathfrak{A}(\mathcal{O}) \neq \mathbb{C} \cdot 1$. Then each of these algebras contains operators with support $\mathbb{R}^4$ in momentum space.*

Proof: The proof of this result is based on two well known facts, cf. for example [10, Sec. VII]. First, for any local operator $A \in \mathfrak{A}(\mathcal{O}_0)$, $A \neq c\,1$, the Fourier transform of $x \rightarrow \alpha_x(A)\Omega$, where Ω is the vacuum vector, has Lorentz invariant support which contains a hyperboloid $H_\mu = \{p \in \mathbb{R}^4 : p^2 = \mu^2, p_0 \geq 0\}$ for some $\mu \geq 0$. Second, if $f, g \in L^1(\mathbb{R}^4)$ are such that $\alpha_f(A)\Omega$ and $\alpha_g(A)\Omega$ are different from 0, then the function $x \rightarrow \alpha_f(A)\alpha_x(\alpha_g(A))\Omega$ does not vanish for large spacelike x, as may be seen from the cluster theorem. Being the boundary value of an analytic function in the forward tube $\mathbb{R}^4 + iV_+$, because of the spectrum condition, it can therefore not vanish on open subsets of $\mathbb{R}^4$.

Now let $q_i, i \in \mathbb{N}$, be a countable dense subset of points on the hyperboloid H_μ and let $\varepsilon_k, k \in \mathbb{N}$, be a sequence of positive numbers tending to 0. For $i, j, k \in \mathbb{N}$ we define open neighbourhoods of the points $(q_i + q_j)$,

$$\Delta_{ijk} \doteq \{p \in \mathbb{R}^4 : |p - q_i - q_j| < \varepsilon_k\}$$

and consider the corresponding sets

$$N_{ijk} = \{x \in \mathbb{R}^4 : E(\Delta_{ijk})A\alpha_x(A)\Omega = 0\},$$

where $E(\,.\,)$ is the spectral resolution of the translations $U \,|\, \mathbb{R}^4$. Making use of the spectrum condition it follows that N_{ijk} either does not contain any open set or coincides with all of $\mathbb{R}^4$. The latter possibility can be excluded since it would imply that $E(\Delta_{ijk})\alpha_{x'}(A)\alpha_{x''}(A)\Omega = 0$ for all $x', x'' \in \mathbb{R}^4$. From this it would

in turn follow that for any choice of functions $f_i, f_j \in L^1(\mathbb{R}^4)$ whose Fourier transforms have support in a ball of radius $\varepsilon_k/3$ about q_i and q_j, respectively, there holds

$$\alpha_{f_i}(A)\alpha_x(\alpha_{f_j}(A))\Omega = E(\Delta_{ijk})\alpha_{f_i}(A)\alpha_x(\alpha_{f_j}(A))\Omega = 0$$

for all $x \in \mathbb{R}^4$. But this is incompatible with the facts mentioned above. Hence each N_{ijk} is a closed nowhere dense subset of $\mathbb{R}^4$, or empty. Since the countable union of such sets is meager (of first category) its complement is dense in $\mathbb{R}^4$ and there exists in each neighbourhood of the origin in $\mathbb{R}^4$ a point y which is not contained in any one of the sets N_{ijk}. Hence, setting $B = A\alpha_y(A)$, there holds

$$E(\Delta_{ijk})B\Omega \neq 0 \quad \text{for all} \quad i,j,k \in \mathbb{N}.$$

Next we consider the sets

$$N_{ijk,lmn} = \{x \in \mathbb{R}^4 : E(\Delta_{ijk})B\alpha_x(B^*)E(\Delta_{lmn}) = 0\}.$$

Multiplying the operator in the defining condition of this set from the right by $U(x)$ we see, using once more the spectrum condition, that also this set cannot contain any open subset, unless it coincides with $\mathbb{R}^4$. The latter possibility can again be excluded since it would imply, by an application of the mean ergodic theorem to the unitary group $U \,|\, \mathbb{R}^4$, that $E(\Delta_{ijk})BE(\{0\})B^*E(\Delta_{lmn}) = 0$, which would be in conflict with the properties of B established before. Hence also the sets $N_{ijk,lmn}$ are closed and nowhere dense. So again there exist points z in any neighbourhood of the origin which are not contained in any one of these sets,

$$E(\Delta_{ijk})B\alpha_z(B^*)E(\Delta_{lmn}) \neq 0 \quad \text{for all} \quad i,j,k,l,m,n \in \mathbb{N}.$$

It follows from this result that the operator $C = B\alpha_z(B^*)$ has support in momentum space which contains all points $(q_i + q_j - q_l - q_m)$. For if $f \in L^1(\mathbb{R}^4)$ is a function whose Fourier transform $\widetilde{f}$ is equal to 1 in any given neighbourhood of such a point and equal to 0 in the complement of any slightly larger region there holds

$$E(\Delta_{ijk})\alpha_f(C)E(\Delta_{lmn}) = E(\Delta_{ijk})CE(\Delta_{lmn}) \neq 0$$

for sufficiently large k, n, and consequently $\alpha_f(C) \neq 0$. Since the set of points $(q_i + q_j - q_l - q_m)$ is dense in $\mathbb{R}^4$ this implies that C has support $\mathbb{R}^4$. We finally recall that the points y, z in the construction of C from the original operator $A \in \mathfrak{A}(\mathcal{O}_0)$ can be chosen as close to 0 as one wishes. Hence there exists for any region $\mathcal{O}$ which contains the closure of $\mathcal{O}_0$ in its interiour an operator $C \in \mathfrak{A}(\mathcal{O})$ with the stated support properties in momentum space. $\square$

Whereas local operators cannot have, in the strict mathematical sense, compact support in momentum space, it is an important consequence of their continuity properties with respect to translations that they have "almost compact support", as will be explained now.

Lemma 2.2 *Let the local algebras be non-trivial and let $\mathcal{O} \subset \mathbb{R}^4$ be given.*
(i) For any $A \in \mathfrak{A}(\mathcal{O})$ and $\varepsilon > 0$ there exists a closed ball $\widetilde{\mathcal{O}} \subset \mathbb{R}^4$ centered at 0 and an operator $\widetilde{A} \in \widetilde{\mathfrak{A}}(\widetilde{\mathcal{O}})$ such that $||A - \widetilde{A}|| \leq \varepsilon ||A||$.
(ii) Conversely, given any such ball $\widetilde{\mathcal{O}}$ there exist an operator $A \in \mathfrak{A}(\mathcal{O})$ and operators $\widetilde{A}_\mu, \mu \geq 1$, with support $\mu\widetilde{\mathcal{O}}$ in momentum space such that $||A - \widetilde{A}_\mu|| \leq \mu^{-1}||A||$.

Proof: (i) Let $f \in \mathcal{S}(\mathbb{R}^4)$ be any test function whose Fourier transform has support in some ball $\widetilde{\mathcal{O}}$ about 0, does not vanish in its interiour, and satisfies $\int dx\, f(x) = 1$. Setting $f_\mu(x) \doteq \mu^4 f(\mu x), \mu \geq 1$, we define operators $\widetilde{A}_\mu \doteq \alpha_{f_\mu}(A)$ which are, by construction, elements of $\widetilde{\mathfrak{A}}(\mu\widetilde{\mathcal{O}})$. Moreover, there holds $||A - \widetilde{A}_\mu|| \leq \int dx\, |f(x)|\, ||\alpha_{\mu^{-1}x}(A) - A||$. In view of the continuity properties of A with respect to translations, the right hand side of this inequality can be made arbitrarily small for sufficiently large μ.

(ii) According to the preceding lemma there exist local operators $A \in \mathfrak{A}(\mathcal{O})$ which have support $\mathbb{R}^4$ in momentum space. Moreover, there also exist such operators for which the corresponding operator functions $x \rightarrow \alpha_x(A)$ are differentiable in norm. (Since all of these functions are norm continuous, the differentiability can be accomplished by convolution with suitable test functions whose Fourier transforms do not vanish.) We pick any such operator A and consider the corresponding operators $\widetilde{A}_\mu$ introduced in the preceding step. Because of the support properties of A in momentum space the operators $\widetilde{A}_\mu$ have support $\mu\widetilde{\mathcal{O}}$ and because of the differentiability of $x \rightarrow \alpha_x(A)$ there holds $||\alpha_x(A) - A|| \leq c_A\, |x|$, $x \in \mathbb{R}^4$. Hence it follows from the estimate given in the preceding step that $||A - \widetilde{A}_\mu|| \leq \mu^{-1} c_{A,f}$, where $c_{A,f}$ is a positive constant depending on A and f. If $c_{A,f} \leq ||A||$ the statement follows. Otherwise we consider the operator $A' \doteq \zeta 1 + \eta A$, where $\eta = {c_{A,f}}^{-1}||A||$ and the number ζ is chosen in such a way that $||A'|| = ||A||$. This operator is still an element of $\mathfrak{A}(\mathcal{O})$ and the corresponding operators $\widetilde{A}'_\mu \doteq \zeta 1 + \eta \widetilde{A}_\mu$ still have support $\mu\widetilde{\mathcal{O}}$ in momentum space since a non-zero c-number factor does not change the support properties of an operator and the unit operator has support $\{0\}$. It thus follows from the preceding estimate that $||A' - \widetilde{A}'_\mu|| \leq \mu^{-1}||A'||$, which completes the proof. □

Although the size of the regions $\mathcal{O}$, $\widetilde{\mathcal{O}}$ in the second part of this statement is in principle arbitrary, it is clear that the respective operators A will be close in norm to the identity (i.e., the constant η in the last step of the above argument will be very small) if the regions do not comply with the uncertainty principle. In order to obtain operators which do not have a dominant c-number part $\zeta 1$, the product of the diameters of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ should be of the order of Plancks constant. It is conceivable that in some theories fitting into our general setting the latter condition does not suffice to ensure the existence of non-trivial ("quantum") operators for arbitrarily small regions $\mathcal{O}$. The orbit of any local operator under the action of the renormalization group would then tend in norm to a multiple of the identity, hence the quantum correlations between "comparable observables" would disappear in the scaling limit. This possibility will be discussed in Sec. 4 in more detail.

3 Scaling algebra and renormalization group

In this section we introduce the concept of scaling algebra which allows us to express the basic ideas of the renormalization group in the algebraic setting. Our starting point is a local, Poincaré covariant net $\mathfrak{A}, \alpha_{\mathcal{P}_+^\uparrow}$ as described in Sec. 2, which is assumed to be given. It is called the underlying net or underlying theory.

To fix ideas we assume that the net $\mathfrak{A}, \alpha_{\mathcal{P}_+^\uparrow}$ is defined at spatio-temporal scale $\lambda = 1$. As was discussed in the Introduction, one then obtains the corresponding nets at any other scale $\lambda > 0$ by setting

$$\mathcal{O} \to \mathfrak{A}_\lambda(\mathcal{O}) \doteq \mathfrak{A}(\lambda \mathcal{O}) \,. \tag{3.1}$$

The Poincaré transformations at scale λ are given by

$$\alpha^{(\lambda)}_{\Lambda,x} \doteq \alpha_{\Lambda,\lambda x} \,, \quad (\Lambda, x) \in \mathcal{P}_+^\uparrow \,. \tag{3.2}$$

Note that $\mathfrak{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow}$ defines again a local, Poincaré covariant net over Minkowski space which complies with the conditions given in Sec. 2. Thus, in accord with the procedure in the conventional field theoretical setting, we keep Minkowski space fixed and interpret the properties of the underlying theory at small scales λ in terms of the modified theories (nets) $\mathfrak{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow}$.

It is apparent that the nets $\mathfrak{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow}$ describe, for different values of λ, in general distinct theories (with different energy-momentum spectrum, collision cross sections, "running" coupling constants etc.). Within the algebraic setting these differences find there formal expression in the fact that the corresponding nets are non-isomorphic. Conversely, any two local, Poincaré-covariant nets which are isomorphic are physically indistinguishable and consequently represent the same theory. We recall the notion of net isomorphism in the following definition.

Definition: For $j = 1, 2$, let $\mathcal{O} \to \mathfrak{A}^{(j)}(\mathcal{O})$, $\alpha^{(j)}_{\mathcal{P}_+^\uparrow}$ be two local, Poincaré covariant nets on Minkowski space with C^*-inductive limits $\mathfrak{A}^{(j)}$. The two nets are said to be *isomorphic* if there is an isomorphism $\phi : \mathfrak{A}^{(1)} \to \mathfrak{A}^{(2)}$ which preserves localization,

$$\phi(\mathfrak{A}^{(1)}(\mathcal{O})) = \mathfrak{A}^{(2)}(\mathcal{O}) \,, \quad \mathcal{O} \subset \mathbb{R}^4$$

and intertwines the Poincaré transformations,

$$\phi \circ \alpha^{(1)}_{\Lambda,x} = \alpha^{(2)}_{\Lambda,x} \circ \phi \,, \quad (\Lambda, x) \in \mathcal{P}_+^\uparrow \,.$$

Any such isomorphism ϕ is called a *net isomorphism.* A net isomorphism which maps a given net onto itself is called an *internal symmetry.*

According to this definition the nets $\mathfrak{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow}$ are isomorphic for different values of λ if and only if dilations are a (geometrical) symmetry of the underlying theory. The physical content of the theory is then invariant under changes of the

spatio-temporal scale. We are here primarily interested in those cases where the underlying theory does *not* possess such a symmetry. Consequently we cannot rely on the notion of net-isomorphism in order to compare the properties of the theory at different scales.

This comparison can be accomplished, however, with the help of the renormalization group transformations R_λ, considered in the Introduction, which map the underlying net $\mathfrak{A}$ onto the nets $\mathfrak{A}_\lambda$ at any other scale (though without preserving algebraic relations). As was explained, the transformations R_λ are used in the conventional setting of quantum field theory to identify particular observables at different scales. Yet such a detailed information is not necessary for the interpretation of the short distance properties of a theory. All what matters is that renormalization group transformations do not change the fundamental physical units c and $\hbar$ and are continuous, cf. conditions (i), (ii) and (iii) in the Introduction. As we shall see, it is not even necessary for the short distance analysis to specify the transformations R_λ explicitly. It suffices to have control on the phase space properties of the orbits $\lambda \to R_\lambda(A)$, $\lambda > 0$, of local observables $A \in \mathfrak{A}$ under these transformations. Thus, instead of starting from some *ad hoc* choice of the transformation R_λ, we will consider the set of all operator functions of the scaling parameter λ with values in $\mathfrak{A}$ which exhibit the relevant features of the orbits of observables under the action of the renormalization group. These functions will constitute the elements of the scaling algebra.

Before we can turn to the actual definition of this algebra we have to elaborate the relevant properties of the functions $\lambda \to R_\lambda(A)$, $A \in \mathfrak{A}$. Given any $A \in \mathfrak{A}(\mathcal{O})$ it follows from condition (i) in the Introduction that

$$R_\lambda(A) \in \mathfrak{A}_\lambda(\mathcal{O}) = \mathfrak{A}(\lambda\mathcal{O})\,, \quad \lambda > 0\,. \tag{3.3}$$

Thus these functions have specific localization properties in configuration space. Next, let us determine the properties of $R_\lambda(A)$ in momentum space. As was explained in Sec. 2, local operators $A \in \mathfrak{A}(\mathcal{O})$ cannot have compact support in momentum space, but they have almost compact support in the sense of part (i) of Lemma 2.2. More precisely, given $\delta > 0$, there is a compact set $\widetilde{\mathcal{O}} \subset \mathbb{R}^4$ and an operator $\widetilde{A} \in \widetilde{\mathfrak{A}}(\widetilde{\mathcal{O}})$ such that $||\, A - \widetilde{A}\,|| < \delta\, ||\, A\,||$. According to condition (ii) there holds $R_\lambda(\widetilde{A}) \in \widetilde{\mathfrak{A}}_\lambda(\widetilde{\mathcal{O}}) = \widetilde{\mathfrak{A}}(\lambda^{-1}\widetilde{\mathcal{O}})$. Thus, making use of the continuity properties of R_λ, cf. condition (iii), we conclude (choosing δ sufficiently small) that for any $\varepsilon > 0$ there is some compact set $\widetilde{\mathcal{O}}$ such that

$$R_\lambda(A) \in \widetilde{\mathfrak{A}}(\lambda^{-1}\widetilde{\mathcal{O}}) + \varepsilon\, \mathfrak{U}\,, \quad \lambda > 0 \tag{3.4}$$

where $\mathfrak{U}$ is the unit ball in $\mathfrak{A}$. Hence in this approximate sense the orbits of local observables under renormalization group transformations have characteristic support properties in momentum space as well. Roughly speaking, the volume of phase space occupied by these operators is independent of the scale λ.

It will greatly simplify our discussion that the mildly cumbersome relation (3.4) can be replaced by a continuity condition with respect to space-time translations, as is shown in the subsequent lemma.

Lemma 3.1 *Let $\lambda \rightarrow A_\lambda$, $\lambda > 0$, be a function with values in $\mathfrak{A}$ which is uniformly bounded in norm. Then the following two statements are equivalent.*
(i) For each $\varepsilon > 0$ there exists a compact set $\widetilde{\mathcal{O}} \subset \mathbb{R}^4$ such that $A_\lambda \in \widetilde{\mathfrak{A}}(\lambda^{-1}\widetilde{\mathcal{O}}) + \varepsilon\,\mathfrak{U}$ for all $\lambda > 0$.
(ii) $\sup_{\lambda>0} \| \alpha_{\lambda x}(A_\lambda) - A_\lambda \| \rightarrow 0$ for $x \rightarrow 0$.

Proof: That the first statement implies the second one may be seen as follows. Let $\varepsilon > 0$ and let $\widetilde{A}_\lambda \in \widetilde{\mathfrak{A}}(\lambda^{-1}\widetilde{\mathcal{O}})$ be such that $\| A_\lambda - \widetilde{A}_\lambda \| \leq \varepsilon$, $\lambda > 0$, and consequently

$$\sup_{\lambda>0} \| \alpha_{\lambda x}(A_\lambda) - A_\lambda \| \leq \sup_{\lambda>0} \| \alpha_{\lambda x}(\widetilde{A}_\lambda) - \widetilde{A}_\lambda \| + 2\varepsilon\,.$$

Since A_λ is uniformly bounded for $\lambda > 0$ the same holds true for $\widetilde{A}_\lambda$. We pick any function $f \in L^1(\mathbb{R}^4)$ whose Fourier transform is equal to 1 in a neighbourhood of $\widetilde{\mathcal{O}}$ and set $f_\lambda(x) \doteq \lambda^{-4} f(\lambda^{-1}x)$. Because of the support properties of $\widetilde{A}_\lambda$ in momentum space we have $\widetilde{A}_\lambda = \alpha_{f_\lambda}(\widetilde{A}_\lambda)$, hence

$$\sup_{\lambda} \| \alpha_{\lambda x}(\widetilde{A}_\lambda) - \widetilde{A}_\lambda \| \leq \int dy\, |f(y-x) - f(y)| \cdot \sup_{\lambda>0} \| \widetilde{A}_\lambda \|\,.$$

Since the right hand side of this inequality vanishes for $x \rightarrow 0$, the second statement follows.

For the proof of the converse direction we put $\eta(x) \doteq \sup_{\lambda>0} \| \alpha_{\lambda x}(A_\lambda) - A_\lambda \|$. By assumption this function is bounded and vanishes for $x \rightarrow 0$. Given $\varepsilon > 0$, we can therefore find some function $g \in L^1(\mathbb{R}^4)$ such that $\int dx\, g(x) = 1$, $\int dx\, |g(x)| \eta(x) \leq \varepsilon$ and the Fourier transform of g has compact support in some (sufficiently large) region $\widetilde{\mathcal{O}}$. Proceeding to the scaled functions g_λ as above it follows that the operators $\widetilde{A}_\lambda \doteq \alpha_{g_\lambda}(A_\lambda)$ have support in momentum space in the region $\lambda^{-1}\widetilde{\mathcal{O}}$. Since $\int dx\, g_\lambda(x) = 1$ there holds also $\| A_\lambda - \widetilde{A}_\lambda \| \leq \int dx\, |g(x)| \eta(x) \leq \varepsilon$ for $\lambda > 0$, proving the first statement. □

In view of this lemma and the fact that the functions $\lambda \rightarrow R_\lambda(A)$ are uniformly bounded, cf. condition (iii) in the Introduction, we can proceed from relation (3.4) to the equivalent statement

$$\sup_{\lambda>0} \| \alpha_{\lambda x}(R_\lambda(A)) - R_\lambda(A) \| \rightarrow 0 \quad \text{for} \quad x \rightarrow 0\,. \tag{3.5}$$

Taking into account that angular momentum has the dimension of $\hbar$ and that renormalization group transformations do not change the unit of action one is led by a similar spectral analysis of the functions $\lambda \rightarrow R_\lambda(A)$ with respect to the Lorentz transformations $\alpha_{\mathcal{L}_+^\uparrow}$ to the conclusion that also

$$\sup_{\lambda>0} \| \alpha_\Lambda(R_\lambda(A)) - R_\lambda(A) \| \rightarrow 0 \quad \text{for} \quad \Lambda \rightarrow 1,\ \Lambda \in \mathcal{L}_+^\uparrow\,. \tag{3.6}$$

We regard the properties (3.3), (3.5) and (3.6) as the distinctive features of the orbits of local operators under renormalization group transformations. As a matter

of fact one can give arguments that any function $\lambda \to A_\lambda$ with these properties arises from some transformation which satisfies conditions (i), (ii) and (iii) in the Introduction with arbitrary precision. We may therefore forget the underlying transformations and consider the set of *all* functions $\lambda \to A_\lambda$ with these properties. Given any such function we interpret its values A_λ as elements of the respective nets (theories) $\mathfrak{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow}$. Thus these functions provide information as to which operators are identified at different scales. Since the operators A_λ are only restricted by phase space conditions, this view may appear to be problematic if one thinks of the rigid assignment of operators at different scales in the conventional approach to the renormalization group. But let us recall that according to the algebraic point of view the physical information of a theory is contained in (and can be recovered from) the net. It is not necessary to fix the interpretation of individual operators. Phrased differently: the physically relevant information about the action of the renormalization group is contained in the particular embedding of the nets $\mathfrak{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow}$ at different scales λ into each other, which is provided by the functions $\lambda \to A_\lambda$. It is *not* contained in the individual orbits.

We emphasize that this approach provides a redundant description of the renormalization group since the orbits induced by all admissible transformations R_λ are taken into consideration. This causes some minor technical complications which can be handled, however, by standard functional analytic techniques. What one gains on the other hand by this method is a general formalism for the renormalization group analysis which is canonically associated with any given theory and free of arbitrariness. Moreover, it circumvents the actual construction of the renormalization group transformations R_λ.

To stress the latter point let us note that functions $\lambda \to A_\lambda$ with the desired properties can easily be constructed in abundance. This follows from the second part of Lemma 2.2. According to that statement there exist for fixed regions $\mathcal{O}, \widetilde{\mathcal{O}}$ and any $\lambda > 0$ operators $A_\lambda \in \mathfrak{A}(\lambda\mathcal{O})$, $|| A_\lambda || = 1$, and $\widetilde{A}_{\lambda,\mu} \in \widetilde{\mathfrak{A}}(\mu\lambda^{-1}\widetilde{\mathcal{O}})$, $\mu \geq 1$, such that $|| A_\lambda - \widetilde{A}_{\lambda,\mu} || \leq \mu^{-1}$. Thus the corresponding functions $\lambda \to A_\lambda$ satisfy conditions (3.3) and (3.4), hence also (3.5). In order to satisfy also condition (3.6) it suffices to average the operators A_λ, $\lambda > 0$ over a fixed, arbitrarily small neighbourhood $\mathcal{N}$ of unity in $\mathcal{L}_+^\uparrow$, $\overline{A}_\lambda \doteq \int_\mathcal{N} d\Lambda \, \alpha_\Lambda(A_\lambda)$, where $d\Lambda$ is the Haar-measure. The resulting functions $\lambda \to \overline{A}_\lambda$ then have all desired properties.

After this motivation of our particular approach to the renormalization group analysis let us turn now to the definition of the scaling algebras. As was explained, we consider functions $\underline{A} : \mathbb{R}^+ \to \mathfrak{A}$ from the domain $\mathbb{R}^+$ of the scaling parameter λ to the underlying algebra of observables. (In order to distinguish these functions from elements of $\mathfrak{A}$ we mark them in the following by underlining.) Since we want to interpret the values $\underline{A}_\lambda$ of these functions as elements of the local, covariant nets $\mathfrak{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow}$, it is natural to induce the following algebraic structures: given any

two functions $\underline{A}, \underline{B}$ and $a, b \in \mathbb{C}$ we set for $\lambda > 0$

$$\begin{aligned} (a\underline{A} + b\underline{B})_\lambda &\doteq a\underline{A}_\lambda + b\underline{B}_\lambda \\ (\underline{A} \cdot \underline{B})_\lambda &\doteq \underline{A}_\lambda \cdot \underline{B}_\lambda \\ (\underline{A}^*)_\lambda &\doteq \underline{A}_\lambda^* \,. \end{aligned} \tag{3.7}$$

Thereby the functions $\underline{A}$ acquire the structure of a unital *-algebra, the unit being given by $\underline{1}_\lambda = 1$. Moreover, since we are only interested in uniformly bounded functions it is natural to introduce the norm

$$|| \, \underline{A} \, || \doteq \sup_{\lambda > 0} || \, \underline{A}_\lambda \, || \tag{3.8}$$

which in fact is a C^*-norm. Bearing in mind that $\alpha^{(\lambda)}_{\Lambda,x} = \alpha_{\Lambda,\lambda x}$, the induced action of the Poincaré transformations on the functions is given by

$$(\underline{\alpha}_{\Lambda,x}(\underline{A}))_\lambda \doteq \alpha_{\Lambda,\lambda x}(\underline{A}_\lambda) \,. \tag{3.9}$$

It follows that the continuity requirements (3.5) and (3.6) can be expressed in the simple form

$$|| \, \underline{\alpha}_{\Lambda,x}(\underline{A}) - \underline{A} \, || \to 0 \quad \text{for} \quad (\Lambda, x) \to (1, 0) \,. \tag{3.10}$$

It remains to impose on the functions the localization condition (3.3). This is accomplished with the help of the following definition.

Definition: Let $\mathcal{O} \subset \mathbb{R}^4$ be any open, bounded region. Then $\underline{\mathfrak{A}}(\mathcal{O})$ denotes the set of all uniformly bounded functions $\underline{A}$ which are continuous with respect to Poincaré transformations in the sense of relation (3.10) and satisfy

$$\underline{A}_\lambda \in \mathfrak{A}(\lambda \mathcal{O}) \,, \quad \lambda > 0 \,.$$

Since each $\mathfrak{A}(\lambda\mathcal{O})$ is a C^*-algebra it follows that $\underline{\mathfrak{A}}(\mathcal{O})$ is a C^*-algebra as well: it is stable under the algebraic operations (3.7) and complete with respect to the C^*-norm (3.8). (Note that $|| \, \underline{\alpha}_{\Lambda,x}(\underline{A}) \, || = || \, \underline{A} \, ||$, hence the limit of any Cauchy sequence whose elements satisfy the continuity condition (3.10) again satisfies this condition.) It is also apparent from the definition that $\underline{\mathfrak{A}}(\mathcal{O})$ is monotonous with respect to $\mathcal{O}$,

$$\underline{\mathfrak{A}}(\mathcal{O}_1) \subset \underline{\mathfrak{A}}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2 \,. \tag{3.11}$$

Thus the assignment $\mathcal{O} \to \underline{\mathfrak{A}}(\mathcal{O})$ defines a net of C^*-algebras over Minkowski space. Since $\mathcal{O}_1 \subset \mathcal{O}_2'$ implies that also $\lambda\mathcal{O}_1 \subset (\lambda\mathcal{O}_2)'$, $\lambda > 0$ (the velocity of light, c, is kept constant at each scale) it follows from the locality of the underlying theory that the net $\underline{\mathfrak{A}}$ is local, too,

$$\underline{A}\,\underline{B} = \underline{B}\,\underline{A} \quad \text{for} \quad \underline{A} \in \underline{\mathfrak{A}}(\mathcal{O}_1), \underline{B} \in \underline{\mathfrak{A}}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2' \,. \tag{3.12}$$

Finally, because of the equality of sets $(\Lambda\lambda\mathcal{O}+\lambda x) = \lambda(\Lambda\mathcal{O}+x)$, $\lambda > 0$, it follows from (3.9) and covariance of the underlying theory that

$$\underline{\alpha}_{\Lambda,x}(\underline{\mathfrak{A}}(\mathcal{O})) = \underline{\mathfrak{A}}(\Lambda\mathcal{O}+x)\,. \tag{3.13}$$

Hence the Poincaré transformations $\underline{\alpha}_{\mathcal{P}_+^\uparrow}$ are automorphisms of the net $\underline{\mathfrak{A}}$. We have thus arrived at a mathematical setting which is similar to our starting point.

Definition: The local, covariant net $\underline{\mathfrak{A}}, \underline{\alpha}_{\mathcal{P}_+^\uparrow}$ is called *scaling net* of the underlying theory. The C^*-inductive limit of all local algebras $\underline{\mathfrak{A}}(\mathcal{O})$ is called *scaling algebra* and denoted by $\underline{\mathfrak{A}}$. (As is common practice, we denote the net and the corresponding global C^*-algebra by the same symbol.)

Remark: The elements of the global scaling algebra $\underline{\mathfrak{A}}$ can still be represented by certain specific functions $\lambda \to \underline{A}_\lambda$ with values in $\mathfrak{A}$.

It is straightforward to describe in this formalism a change of the spatio-temporal scale. Such changes are induced by an automorphic action $\underline{\sigma}_{\mathbb{R}^+}$ of the multiplicative group $\mathbb{R}^+$ on the scaling algebra $\underline{\mathfrak{A}}$, given for any $\mu \in \mathbb{R}^+$ by

$$(\underline{\sigma}_\mu(\underline{A}))_\lambda \doteq \underline{A}_{\mu\lambda}\,, \quad \lambda > 0\,. \tag{3.14}$$

As is easily verified, there holds

$$\underline{\sigma}_\mu(\underline{\mathfrak{A}}(\mathcal{O})) = \underline{\mathfrak{A}}(\mu\mathcal{O})\,, \quad \mathcal{O} \subset \mathbb{R}^4 \tag{3.15}$$

and

$$\underline{\sigma}_\mu \circ \underline{\alpha}_{\Lambda,x} = \underline{\alpha}_{\Lambda,\mu x} \circ \underline{\sigma}_\mu\,, \quad (\Lambda, x) \in \mathcal{P}_+^\uparrow\,. \tag{3.16}$$

According to our previous discussion the scaling transformations $\underline{\sigma}_{\mathbb{R}^+}$ are to be viewed as renormalization group transformations which relate the observables at different scales. They appear in the present setting as geometrical symmetries; this greatly simplifies their analysis.

The scaling algebra $\underline{\mathfrak{A}}$ comprises the orbits of local observables under all possible renormalization group transformations which comply with the basic conditions given in the Introduction. So in this sense it is maximal. In applications of the formalism it may sometimes be convenient to impose further constraints on the orbits which amounts to proceeding to subnets of $\underline{\mathfrak{A}}$. For example, it seems natural to restrict attention to those elements of $\underline{\mathfrak{A}}$ on which the scaling transformations $\underline{\sigma}_\mu$, $\mu > 0$, act norm continuously. Let us note, however, that continuity at $\mu = 0$ (from the right) would be too strong a requirement. For only multiples of the identity $\underline{1}$ have this property. In the present investigation we will not impose any further constraints on the functions $\underline{A}$ and work with the maximal net $\underline{\mathfrak{A}}$.

It is our objective to study the properties of the physical states of the underlying theory at arbitrarily small scales. To this end we will lift these states to the scaling algebra and study their behaviour under scaling transformations. Before

we explain this procedure we introduce some notation. Let $\underline{\omega}$ be any state on the scaling algebra $\underline{\mathfrak{A}}$ and let $(\underline{\pi}, \underline{\mathcal{H}}, \underline{\Omega})$ be the corresponding GNS-representation. The kernel of $\underline{\pi}$ (i.e., the closed two-sided ideal consisting of all elements $\underline{A} \in \underline{\mathfrak{A}}$ for which $\underline{\pi}(\underline{A}) = 0$) will be denoted by $\ker(\underline{\pi})$ and the quotient of $\underline{\mathfrak{A}}$ with respect to $\ker(\underline{\pi})$ by

$$\underline{\mathfrak{A}}^{\pi} \doteq \underline{\mathfrak{A}}/\ker(\underline{\pi}) \simeq \underline{\pi}(\underline{\mathfrak{A}}) \,. \tag{3.17}$$

Here the symbol $\simeq$ indicates the well-known fact [11, Cor. 1.8.3] that the respective C^*-algebras are isomorphic. The canonical isomorphism ψ between these algebras is given by $\psi(\underline{A}^{\pi}) = \underline{\pi}(\underline{A})$, where $\underline{A}^{\pi}$ is the class of $\underline{A} \in \underline{\mathfrak{A}}$ modulo $\ker(\underline{\pi})$. The projection of $\underline{\omega}$ to the quotient $\underline{\mathfrak{A}}^{\pi}$ is denoted by $\operatorname{proj}\underline{\omega}$ and given by

$$\operatorname{proj}\underline{\omega} \doteq (\underline{\Omega}, \psi(\,.\,)\underline{\Omega}) \,. \tag{3.18}$$

The physical interpretation of the states $\underline{\omega}$ will be based on their projections $\operatorname{proj}\underline{\omega}$, regarded as states on the net

$$\mathcal{O} \to \underline{\mathfrak{A}}^{\pi}(\mathcal{O}) \doteq \underline{\mathfrak{A}}(\mathcal{O})/\ker(\underline{\pi}) \tag{3.19}$$

on Minkowski space. These nets are again local. Moreover, if $\ker(\underline{\pi})$ is invariant under the Poincaré transformations $\underline{\alpha}_{\mathcal{P}_+^\uparrow}$ one can also define an automorphic action of the Poincaré group on $\underline{\mathfrak{A}}^{\pi}$, setting for $(\Lambda, x) \in \mathcal{P}_+^\uparrow$

$$\alpha^{\pi}_{\Lambda,x}(\underline{A}^{\pi}) \doteq (\underline{\alpha}_{\Lambda,x}(\underline{A}))^{\pi} \,, \quad \underline{A}^{\pi} \in \underline{\mathfrak{A}}^{\pi} \,. \tag{3.20}$$

In this way any suitable state $\underline{\omega}$ on $\underline{\mathfrak{A}}$ determines a local, covariant net $\underline{\mathfrak{A}}^{\pi}, \alpha^{\pi}_{\mathcal{P}_+^\uparrow}$ and a distinguished state $\operatorname{proj}\underline{\omega}$ on $\underline{\mathfrak{A}}^{\pi}$.

Given a state ω on the underlying net $\mathfrak{A}, \alpha_{\mathcal{P}_+^\uparrow}$ we intend to define its canonical lift $\underline{\omega}$ to the scaling algebra $\underline{\mathfrak{A}}$ in such a way that we can recover from it by the above procedure the underlying net and state. This leads us to the following definition.

Definition: Let ω be a state on the underlying global algebra $\mathfrak{A}$. Its *canonical lift* $\underline{\omega}$ on the scaling algebra $\underline{\mathfrak{A}}$ is defined by

$$\underline{\omega}(\underline{A}) \doteq \omega(\underline{A}_{\lambda=1}) \,, \quad \underline{A} \in \underline{\mathfrak{A}} \,.$$

It is apparent that $\underline{\omega}$ is a state on $\underline{\mathfrak{A}}$ and we shall see with the help of the subsequent lemma that it provides the desired information.

Lemma 3.2 *Let ω be a state on $\mathfrak{A}$, let $(\pi, \mathcal{H}, \Omega)$ be the corresponding GNS-representation and let $\underline{\omega}$ be the canonical lift of ω on $\underline{\mathfrak{A}}$ with corresponding GNS-representation $(\underline{\pi}, \underline{\mathcal{H}}, \underline{\Omega})$. There exists a unitary $W : \underline{\mathcal{H}} \to \mathcal{H}$ such that $W\underline{\Omega} = \Omega$ and*

$$W\underline{\pi}(\underline{A})W^{-1} = \pi(\underline{A}_{\lambda=1}) \,, \quad \underline{A} \in \underline{\mathfrak{A}} \,.$$

Moreover, for each bounded region $\mathcal{O}$ it holds that

$$W \underline{\pi}(\underline{\mathfrak{A}}(\mathcal{O}))W^{-1} = \pi(\mathfrak{A}(\mathcal{O}))\,,$$

i.e., W induces an isomorphism between the C^*-algebras $\underline{\pi}(\underline{\mathfrak{A}})$ and $\pi(\mathfrak{A})$ which preserves localization.

Proof: It follows from the definition of the canonical lift that

$$(\underline{\Omega}, \underline{\pi}(\underline{A})\underline{\Omega}) = (\Omega, \pi(\underline{A}_{\lambda=1})\Omega)\,, \quad \underline{A} \in \underline{\mathfrak{A}}\,.$$

Hence the map $W : \underline{\mathcal{H}} \to \mathcal{H}$ given by

$$W \underline{\pi}(\underline{A})\underline{\Omega} \doteq \pi(\underline{A}_{\lambda=1})\Omega\,, \quad \underline{A} \in \underline{\mathfrak{A}}$$

is densely defined, isometric and linear. Setting $\underline{A} = \underline{1}$ we have in particular $W\underline{\Omega} = \Omega$. In order to show that the range of W is dense in $\mathcal{H}$ we proceed as follows. Given any region $\mathcal{O}$ and any operator $A \in \mathfrak{A}(\mathcal{O})$ we define a function $\underline{A}$, setting $\underline{A}_\lambda \doteq A$ for $\lambda = 1$ and $\underline{A}_\lambda \doteq 0$ for $\lambda \neq 1$. One easily checks that $\underline{A} \in \underline{\mathfrak{A}}(\mathcal{O})$ and consequently $\pi(\mathfrak{A}(\mathcal{O})) \subset \pi(\underline{\mathfrak{A}}_{\lambda=1}(\mathcal{O}))$, in an obvious notation. The converse inclusion holds by construction of $\underline{\mathfrak{A}}(\mathcal{O})$, hence

$$\pi(\underline{\mathfrak{A}}_{\lambda=1}(\mathcal{O})) = \pi(\mathfrak{A}(\mathcal{O}))\,. \tag{3.21}$$

Proceding to the inductive limit $\mathcal{O} \nearrow \mathbb{R}^4$ we see that also $\pi(\underline{\mathfrak{A}}_{\lambda=1}) = \pi(\mathfrak{A})$. This shows that the range of W contains the vectors $\pi(\mathfrak{A})\Omega$ which are dense in $\mathcal{H}$ according to the GNS-construction. Thus W maps $\underline{\mathcal{H}}$ onto $\mathcal{H}$ and hence is invertible. Making use of the defining relation of W we conclude that

$$W \underline{\pi}(\underline{A})W^{-1} = \pi(\underline{A}_{\lambda=1})\,, \quad \underline{A} \in \underline{\mathfrak{A}}\,.$$

Hence $W \cdot W^{-1}$ defines an isomorphism mapping $\underline{\pi}(\underline{\mathfrak{A}})$ onto $\pi(\mathfrak{A})$ which, because of relation (3.21), preserves localization. $\Box$

This lemma has several interesting consequences which we list below. First, let us recall the following standard terminology.

Definition: Let $\mathfrak{B}$ be a C^*-algebra and let β be a representation of the Poincaré group $\mathcal{P}_+^\uparrow$ by automorphisms of $\mathfrak{B}$. A representation $(\pi, \mathcal{H})$ of $\mathfrak{B}$ is said to be covariant with respect to the action of $\beta_{\mathcal{P}_+^\uparrow}$ (briefly: a *covariant representation* of $\mathfrak{B}, \beta_{\mathcal{P}_+^\uparrow}$) if there is a continuous unitary (possibly projective) representation V of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$ such that for $(\Lambda, x) \in \mathcal{P}_+^\uparrow$

$$V(\Lambda, x)\pi(B)V(\Lambda, x)^{-1} = \pi(\beta_{\Lambda,x}(B))\,, \quad B \in \mathfrak{B}\,.$$

A covariant representation $(\pi, \mathcal{H})$ is said to be a *vacuum representation* if $V \,|\, \mathbb{R}^4$ satisfies the relativistic spectrum condition, $sp\, V \,|\, \mathbb{R}^4 \subset \overline{V}_+$, and there is a vector $\Omega \in \mathcal{H}$ which is cyclic for $\pi(\mathfrak{B})$ and invariant under the action of $V(\Lambda, x)$, $(\Lambda, x) \in \mathcal{P}_+^\uparrow$. Finally, a state ω on $\mathfrak{B}$ is called a *vacuum state* if the corresponding GNS-representation $(\pi, \mathcal{H}, \Omega)$ is a vacuum representation where Ω has the just mentioned properties.

Corollary 3.3 *Let ω be a state on $\mathfrak{A}$ and let $\underline{\omega}$ be its canonical lift on $\underline{\mathfrak{A}}$.*
(i) $\underline{\omega}$ is a pure state iff ω is pure.
(ii) The GNS-representation $(\underline{\pi}, \underline{\mathcal{H}}, \underline{\Omega})$ of $\underline{\mathfrak{A}}, \underline{\alpha}_{\mathcal{P}_+^\uparrow}$, induced by $\underline{\omega}$, is covariant iff the GNS-representation $(\pi, \mathcal{H}, \Omega)$ of $\mathfrak{A}, \alpha_{\mathcal{P}_+^\uparrow}$, induced by ω, is covariant. Moreover, if $\underline{U}, U$ denote the representations of $\mathcal{P}_+^\uparrow$ on the respective Hilbert spaces $\underline{\mathcal{H}}, \mathcal{H}$, one obtains

$$\underline{U}(\Lambda, x) = W^{-1} U(\Lambda, x) W \,, \quad (\Lambda, x) \in \mathcal{P}_+^\uparrow$$

where $W : \underline{\mathcal{H}} \to \mathcal{H}$ is the unitary map appearing in the preceding lemma.
(iii) $\underline{\omega}$ is a vacuum state iff ω is a vacuum state.

Proof: These statements are an elementary consequence of Lemma 3.2. We confine ourselves to establishing the second one. Let $(\pi, \mathcal{H}, \Omega)$ be a covariant representation of $\mathfrak{A}, \alpha_{\mathcal{P}_+^\uparrow}$. There hold the equalities (cf. Lemma 3.2)

$$\begin{aligned} W \underline{\pi}(\underline{\alpha}_{\Lambda,x}(\underline{A})) W^{-1} &= \pi((\underline{\alpha}_{\Lambda,x}(\underline{A}))_{\lambda=1}) \\ &= \pi(\alpha_{\Lambda,x}(\underline{A}_{\lambda=1})) = U(\Lambda, x) \pi(\underline{A}_{\lambda=1}) U(\Lambda, x)^{-1} \\ &= U(\Lambda, x) W \underline{\pi}(\underline{A}) W^{-1} U(\Lambda, x)^{-1} \,. \end{aligned}$$

Hence $(\underline{\pi}, \underline{\mathcal{H}}, \underline{\Omega})$ is a covariant representation of $\underline{\mathfrak{A}}, \underline{\alpha}_{\mathcal{P}_+^\uparrow}$, the corresponding unitary representation $\underline{U}$ of $\mathcal{P}_+^\uparrow$ being given by $\underline{U}(\Lambda, x) = W^{-1} U(\Lambda, x) W$, $(\Lambda, x) \in \mathcal{P}_+^\uparrow$. The proof of the "only if" part of the statement is analogous. $\Box$

Let us apply now these results to the vector states ω in the defining (identical) vacuum representation of $\mathfrak{A}, \alpha_{\mathcal{P}_+^\uparrow}$. Since this representation is covariant, the same is true for the representations $(\underline{\pi}, \underline{\mathcal{H}}, \underline{\Omega})$ of $\underline{\mathfrak{A}}, \underline{\alpha}_{\mathcal{P}_+^\uparrow}$ induced by the respective canonical lifts $\underline{\omega}$. As a consequence, the kernels $\ker(\underline{\pi})$ are invariant under the action of $\underline{\alpha}_{\mathcal{P}_+^\uparrow}$. Hence any such vector state ω determines, as outlined above, via its lift $\underline{\omega}$ on $\underline{\mathfrak{A}}$ a local covariant net $\underline{\mathfrak{A}}^{\pi}, \alpha^{\pi}_{\mathcal{P}_+^\uparrow}$ and a state $\mathrm{proj}\, \underline{\omega}$. Recalling that the C^*-algebras $\underline{\mathfrak{A}}^{\pi}$ and $\underline{\pi}(\underline{\mathfrak{A}})$ are canonically isomorphic, it follows from Lemma 3.2 and its corollary that the nets $\underline{\mathfrak{A}}^{\pi}, \alpha^{\pi}_{\mathcal{P}_+^\uparrow}$ and $\mathfrak{A}, \alpha_{\mathcal{P}_+^\uparrow}$ are isomorphic, a net isomorphism ϕ being given by

$$\phi(\underline{A}^{\pi}) = \underline{A}_{\lambda=1} \,, \quad \underline{A}^{\pi} \in \underline{\mathfrak{A}}^{\pi} \,. \tag{3.22}$$

Hence the two nets describe the same physics. Moreover, we have

$$\mathrm{proj}\, \underline{\omega} \circ \phi^{-1} = \omega \,, \tag{3.23}$$

hence with the above choice of ϕ one can also recover from $\mathrm{proj}\, \underline{\omega}$ the underlying state ω.

Let us note that any two isomorphisms between the above nets are related by an internal symmetry transformation of $\mathfrak{A}, \alpha_{\mathcal{P}_+^\uparrow}$. Thus even if one forgets about the specific assignment (3.23) one can recover from $\mathrm{proj}\, \underline{\omega}$ the state ω up to some internal symmetry transformation. The apparent loss of information

about ω is irrelevant since all physical states look alike at very small scales, as we shall see. Moreover, if the internal symmetries of $\mathfrak{A}, \alpha_{\mathcal{P}_+^\uparrow}$ (provided there are any) are not spontaneously broken in the underlying vacuum representation, then the respective vacuum state is invariant under these symmetries. Hence in this relevant case one can recover without ambiguities the properties of the vacuum from its projected lift. These facts show that for the physical interpretation of the states $\mathrm{proj}\,\underline{\omega}$ there is no need to rely on the framework of the underlying theory. This point will be of relevance in our discussion of the scaling limit, where an "underlying theory" has yet to be defined.

Having seen how the physical interpretation of the states ω in the underlying theory can be recovered from their canonical lifts $\underline{\omega}$ it is now straightforward to obtain information about the properties of these states at any other scale $\lambda > 0$. One simply has to compose the lifted states $\underline{\omega}$ with the scaling transformation $\underline{\sigma}_\lambda$,

$$\underline{\omega}_\lambda \doteq \underline{\omega} \circ \underline{\sigma}_\lambda \, . \tag{3.24}$$

Since $\underline{\sigma}_\lambda$ is an automorphism of the scaling algebra $\underline{\mathfrak{A}}$ one can identify the GNS-representation induced by $\underline{\omega}_\lambda$ with $(\underline{\pi}_\lambda, \underline{\mathcal{H}}, \underline{\Omega})$ where

$$\underline{\pi}_\lambda \doteq \underline{\pi} \circ \underline{\sigma}_\lambda \tag{3.25}$$

and $(\underline{\pi}, \underline{\mathcal{H}}, \underline{\Omega})$ is the original representation of $\underline{\mathfrak{A}}$ induced by $\underline{\omega}$. It follows from relation (3.16) and the Poincaré covariance of $(\underline{\pi}, \underline{\mathcal{H}}, \underline{\Omega})$ that $\ker(\underline{\pi}_\lambda)$ is invariant under $\underline{\alpha}_{\mathcal{P}_+^\uparrow}$. Consequently each state $\underline{\omega}_\lambda$ determines a local, covariant net $\underline{\mathfrak{A}}^{\pi_\lambda}, \alpha^{\pi_\lambda}_{\mathcal{P}_+^\uparrow}$ and a state $\mathrm{proj}\,\underline{\omega}_\lambda$ on this net. That this net and state have the desired interpretation is the content of the subsequent proposition.

Proposition 3.4 *Let ω be a vector state in the defining vacuum representation of $\mathfrak{A}$, let $\underline{\omega}$ be its canonical lift on $\underline{\mathfrak{A}}$ and let $\underline{\omega}_\lambda$, $\lambda > 0$, be the corresponding scaled states with GNS-representations $(\underline{\pi}_\lambda, \underline{\mathcal{H}}, \underline{\Omega})$. For each $\lambda > 0$ there exists a net isomorphism ϕ_λ between the nets $\underline{\mathfrak{A}}^{\pi_\lambda}, \alpha^{\pi_\lambda}_{\mathcal{P}_+^\uparrow}$ and $\mathfrak{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow}$ given by*

$$\phi_\lambda(\underline{A}^{\pi_\lambda}) \doteq \underline{A}_\lambda \, , \quad \underline{A}^{\pi_\lambda} \in \underline{\mathfrak{A}}^{\pi_\lambda} \, .$$

Moreover, there holds

$$\mathrm{proj}\,\underline{\omega}_\lambda \circ \phi_\lambda^{-1} = \omega$$

where ω is regarded as a state on the net $\mathfrak{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow}$.

Proof: Let $\sigma_\lambda : \underline{\mathfrak{A}}^{\pi_\lambda} \to \underline{\mathfrak{A}}^{\pi}$ be the map given by

$$\sigma_\lambda(\underline{A}^{\pi_\lambda}) \doteq (\underline{\sigma}_\lambda(\underline{A}))^{\pi} \, , \quad \underline{A}^{\pi_\lambda} \in \underline{\mathfrak{A}}^{\pi_\lambda} \, .$$

Since $\underline{\sigma}_\lambda$ is an automorphism of $\underline{\mathfrak{A}}$ it follows that σ_λ is an isomorphism. Moreover, one obtains

$$\sigma_\lambda(\underline{\mathfrak{A}}^{\pi_\lambda}(\mathcal{O})) = \underline{\mathfrak{A}}^{\pi}(\lambda\mathcal{O}) \, .$$

Noticing that $\phi_\lambda = \phi \circ \sigma_\lambda$, where ϕ is the isomorphism between $\underline{\mathfrak{A}}^\pi, \alpha^\pi_{\mathcal{P}_+^\uparrow}$ and $\underline{\mathfrak{A}}, \alpha_{\mathcal{P}_+^\uparrow}$ defined in relation (3.22), and recalling relation (3.16) it is straightforward to complete the proof of the statement. □

In view of this result it is obvious how the states $\underline{\omega}_\lambda$ on $\underline{\mathfrak{A}}$ can be used to obtain, in the limit $\lambda \searrow 0$, information on the scaling limit of the underlying theory. The discussion of this issue is the subject of the subsequent section.

4 Scaling limit

Let us now investigate the behaviour of physical states ω on $\mathfrak{A}$ under renormalization group transformations in the scaling limit $\lambda \searrow 0$. Hence we consider the family of states on $\underline{\mathfrak{A}}$

$$\underline{\omega}_\lambda \doteq \underline{\omega} \circ \underline{\sigma}_\lambda \,, \quad \lambda > 0, \tag{4.1}$$

as a net directed towards $\lambda = 0$.

Before entering into details, let us point out some at first sight perhaps unexpected mathematical problem appearing in this analysis. As explained, we want to determine the properties of states at small scales with the help of the functions $\underline{A} \in \underline{\mathfrak{A}}$. Now there holds $\bigcap_{\lambda>0} \mathfrak{A}(\lambda\mathcal{O})^- = \mathbb{C}\cdot 1$ (cf. the proof of Lemma 4.1 below), hence any function $\underline{A}$ with the property that $\underline{A}_\lambda$ converges in norm for $\lambda \searrow 0$ inevitably converges to a multiple of the identity. Consequently such functions are not suitable to test the properties of states in the scaling limit since they are not sensitive to their detailed properties; they have the same limit in *every* state on $\mathfrak{A}$. For this reason we did not assume from the outset that the elements of $\underline{\mathfrak{A}}$ are continuous at $\lambda = 0$. As a consequence, the nets $(\underline{\omega}_\lambda)_{\lambda>0}$ are not convergent. To illustrate this fact consider for example the operators $\underline{C}$ of the form $\underline{C}_\lambda = c_\lambda \cdot 1$, $c_\lambda \in \mathbb{C}$, $\lambda > 0$. There holds $\underline{\omega}_\lambda(\underline{C}) = c_\lambda$, hence if not all of the functions $\lambda \rightarrow c_\lambda$ are continuous for $\lambda \searrow 0$ the net $(\underline{\omega}_\lambda)_{\lambda>0}$ does not converge. Thus at this point we pay for the convenience of not having to exhibit any particular renormalization group transformation by admitting as elements of $\underline{\mathfrak{A}}$ all functions which comply with the basic constraints deriving from the renormalization group.

This apparent difficulty can be handled, however, with the help of the Banach-Alaoglu theorem [12], according to which every bounded set in the dual space of a Banach space is pre-compact in the weak-*-topology. Applying this theorem to the family of states $(\underline{\omega}_\lambda)_{\lambda>0}$ on the Banach space $\underline{\mathfrak{A}}$ we see that this family contains (many) subnets which converge in the weak-*-topology for $\lambda \searrow 0$. This leads us to introduce the following terminology.

Definition: Let ω be a state on $\mathfrak{A}$ and $\underline{\omega}$ its canonical lift on $\underline{\mathfrak{A}}$. Each weak-*-limit point of the net $(\underline{\omega}_\lambda)_{\lambda>0}$ for $\lambda \searrow 0$ is called a *scaling limit state* of $\underline{\omega}$. The scaling limit states of $\underline{\omega}$ are denoted by $\underline{\omega}_{0,\iota}$, $\iota \in \mathbb{I}$ where $\mathbb{I}$ is some index set, and the set of all these states is denoted by $SL(\underline{\omega})$.

For the physical interpretation of the scaling limit states $\underline{\omega}_{0,\iota}$ on $\underline{\mathfrak{A}}$ we apply the

same rules as in the case of states at finite scales: we proceed to the local nets

$$\mathcal{O} \to \mathfrak{A}_{0,\iota}(\mathcal{O}) \doteq \underline{\mathfrak{A}}^{\underline{\pi}_{0,\iota}}(\mathcal{O}) = \underline{\mathfrak{A}}(\mathcal{O})/\ker(\underline{\pi}_{0,\iota}) \, , \tag{4.2}$$

where $\underline{\pi}_{0,\iota}$ is the GNS-representation induced by $\underline{\omega}_{0,\iota}$. According to Proposition 3.4 these nets are to be regarded as "limits" of the underlying nets $\mathfrak{A}_\lambda$ at scale $\lambda > 0$, hence our simplified notation. Similarly, the states

$$\omega_{0,\iota} \doteq \operatorname{proj} \underline{\omega}_{0,\iota} \tag{4.3}$$

on $\mathfrak{A}_{0,\iota}$ are limits of the states $\omega_\lambda = \omega$ on the nets $\mathfrak{A}_\lambda$, $\lambda > 0$. Thus by proceeding from the underlying net to the scaling algebra we have been able to define nets and states in the scaling limit $\lambda = 0$. The apparent ambiguities in this construction, which are reflected in the appearance of the index set $\mathbb{I}$, will be discussed below.

We are primarily interested here in the structure of the nets $\mathfrak{A}_{0,\iota}$ and states $\omega_{0,\iota}$ induced by physical states.

Definition: A state ω on the underlying net $\mathfrak{A}$ is said to be a *physical state* if it is locally normal with respect to the underlying vacuum representation (cf. relation (2.8)).

An important ingredient in our short distance analysis is the following lemma, due to Roberts [13], which is based on a result by Wightman [14]. For the convenience of the reader we sketch the proof of this statement.

Lemma 4.1 *Let ω_1, ω_2 be physical states on the net $\mathfrak{A}$. Then for any bounded region $\mathcal{O}$ one has*

$$|| \, (\omega_1 - \omega_2) \, | \, \mathfrak{A}(\lambda\mathcal{O}) \, || \to 0 \quad \text{for} \quad \lambda \to 0 \, ,$$

where the given expression denotes the norm distance of the states, restricted to the respective algebra.

Proof: The proof proceeds in two steps. One first shows that the intersection of the weak closures of all local algebras centered at the origin 0 of Minkowski space, say, is trivial,

$$\bigcap_{\mathcal{O} \ni 0} \mathfrak{A}(\mathcal{O})^- = \mathbb{C} \cdot 1 \, .$$

This is accomplished as follows: if Z is any element in this intersection the same is true for Z^* and, since both operators are localized at 0, there holds $[Z^*, \alpha_x(Z)] = 0$ for (strictly) spacelike x because of locality. But $x \to \alpha_x(Z)$ is continuous in the weak operator topology, hence the commutator vanishes also for lightlike x and $x = 0$. We pick any lightlike vector e and consider the function $\mathbb{R} \ni t \to (\Omega, Z^* \alpha_{te}(Z)\Omega)$, where Ω is the vacuum vector in the defining representation of $\mathfrak{A}$. Its Fourier transform has support in $\mathbb{R}^+ \cup \{0\}$ because of the relativistic spectrum condition. Because of "lightlike commutativity", the function coincides with $t \to (\Omega, \alpha_{te}(Z) Z^* \Omega)$ whose Fourier transform has support in $\mathbb{R}^- \cup \{0\}$. Since the intersection of these two sets is $\{0\}$ and the function is

bounded, it has to be constant. It follows that $U(te)Z\Omega = Z\Omega$ for any $t \in \mathbb{R}$ and all lightlike vectors e. Consequently $U(x)Z\Omega = Z\Omega$ for all $x \in \mathbb{R}^4$ and therefore $Z\Omega = (\Omega, Z\Omega) \cdot \Omega$, by the uniqueness of the vacuum vector. Since Ω is separating[3] for the local algebras $\mathfrak{A}(\mathcal{O})^-$ one arrives at $Z = (\Omega, Z\Omega) \cdot 1$, as claimed.

To complete the proof of the lemma suppose that λ_k, $k \in \mathbb{N}$, is a sequence tending to 0 such that for some $\delta > 0$, $|| (\omega_1 - \omega_2) \,|\, \mathfrak{A}(\lambda_k \mathcal{O}) || \geq \delta$. Then there exist operators $A_k \in \mathfrak{A}(\lambda_k \mathcal{O})$ of norm 1 such that $|\, \omega_1(A_k) - \omega_2(A_k) \,| \geq \delta/2$, $k \in \mathbb{N}$. Since the set of operators A_k, $k \in \mathbb{N}$, is bounded, it is pre-compact in the weak operator topology. Its limit (accumulation) points Z are contained in $\bigcap_{\mathcal{O} \ni 0} \mathfrak{A}(\mathcal{O})^-$ since for any given open region $\mathcal{O}_0$ containing 0 there holds $\mathfrak{A}(\lambda_k \mathcal{O}) \subset \mathfrak{A}(\mathcal{O}_0)^-$ for sufficiently small λ_k, and $\mathfrak{A}(\mathcal{O}_0)^-$ is closed in the weak operator topology. Hence $Z \in \mathbb{C} \cdot 1$. On the other hand, since Z is a weak limit point of the set of operators A_k, $k \in \mathbb{N}$, and the states ω_i, $i = 1, 2$, are locally normal with respect to the vacuum representation, there exists an index l such that $|\, \omega_i(A_l) - \omega_i(Z) \,| \leq \delta/8$, $i = 1, 2$. But this implies that $|\, \omega_1(Z) - \omega_2(Z) \,| \geq \delta/4$ which is a contradiction in view of the fact that Z is a multiple of the identity and states are normalized. □

The following result is an immediate consequence of this lemma. It says that all physical states look alike in the scaling limit.

Corollary 4.2 *Let ω_1, ω_2 be physical states on $\mathfrak{A}$ and let $\underline{\omega}_1$, $\underline{\omega}_2$ be their canonical lifts on $\underline{\mathfrak{A}}$. Then*

$$\lim_{\lambda \searrow 0} || (\underline{\omega}_{1,\lambda} - \underline{\omega}_{2,\lambda}) \,|\, \underline{\mathfrak{A}}(\mathcal{O}) || = 0$$

for all bounded spacetime regions $\mathcal{O}$. In particular, $SL(\underline{\omega}_1) = SL(\underline{\omega}_2)$.

Proof: Let $\mathcal{O}$ be any bounded region and let $\underline{A} \in \underline{\mathfrak{A}}(\mathcal{O})$. By definition

$$(\underline{\omega}_{1,\lambda} - \underline{\omega}_{2,\lambda})(\underline{A}) = \omega_1(\underline{A}_\lambda) - \omega_2(\underline{A}_\lambda) \, , \quad \lambda > 0 \, .$$

But $\underline{A}_\lambda \in \mathfrak{A}(\lambda \mathcal{O})$, so the first part of the statement follows from the preceding lemma. Since the local algebras $\underline{\mathfrak{A}}(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^4$, are norm dense in $\underline{\mathfrak{A}}$, this result implies that $\lim_{\lambda \searrow 0} (\underline{\omega}_{1,\lambda} - \underline{\omega}_{2,\lambda})(\underline{A}) = 0$ for all $\underline{A} \in \mathfrak{A}$. Hence the sets $SL(\underline{\omega}_1)$ and $SL(\underline{\omega}_2)$ coincide. □

In view of the preceding arguments the following remark may be in order: although the limit points of the nets $\underline{A}_\lambda$, $\lambda > 0$, are multiples of the identity, this does not imply that the theory becomes trivial in the scaling limit. For these sequences converge in general only in the weak operator topology. So the limits of products $\underline{A}_\lambda \cdot \underline{B}_\lambda$ need not coincide with the products of the respective limits of the factors $\underline{A}_\lambda$ and $\underline{B}_\lambda$. In this way non-trivial correlations between observables can (and do) persist in the scaling limit.

According to the previous result it suffices for the investigation of the scaling limit theories to concentrate on the analysis of a single physical state. In the subsequent lemma we make use of this fact and exhibit the structure of the scaling limit states inherited from the underlying vacuum state.

[3] A vector Ω is called separating for a set $\mathfrak{B}$ of bounded operators if, for each $B \in \mathfrak{B}$, $B\Omega = 0$ implies $B = 0$.

Lemma 4.3 *Let ω be a physical state on $\mathfrak{A}$ and let $\underline{\omega}$ be its canonical lift on $\underline{\mathfrak{A}}$. Then each scaling limit state $\underline{\omega}_{0,\iota} \in SL(\underline{\omega})$ is a pure, Poincaré invariant vacuum state on $\underline{\mathfrak{A}}$.*

Remark: This statement holds in any number of spacetime dimensions greater than two. In two dimensional spacetime the scaling limit states are also vacuum states, but they need not be pure in general.

Proof: Without restriction of generality we may assume that ω is the underlying vacuum state. In order to prove that the corresponding states $\underline{\omega}_{0,\iota} \in SL(\underline{\omega})$ are pure vacuum states on $\underline{\mathfrak{A}}$ it suffices to show that they are invariant under the action of the Poincaré transformations $\underline{\alpha}_{\mathcal{P}_+^\uparrow}$ and that the respective correlation functions have the right continuity, support (in momentum space) and clustering properties. The proof that the corresponding GNS-representations are vacuum representations is then similar to the one in standard quantum field theory, cf. [15], and may be omitted here.

Let $\underline{\omega}_{0,\iota} \in SL(\underline{\omega})$ and let $(\underline{\omega}_{\lambda_\kappa})_{\kappa\in\mathbb{K}}$ be a subnet of $(\underline{\omega}_\lambda)_{\lambda>0}$ which converges in the weak-*-topology to $\underline{\omega}_{0,\iota}$, i.e., $\lim_\kappa \underline{\omega}_{\lambda_\kappa}(\underline{A}) = \underline{\omega}_{0,\iota}(\underline{A})$ for all $\underline{A} \in \underline{\mathfrak{A}}$. Since each $\underline{\omega}_\lambda$ is a vacuum state on $\underline{\mathfrak{A}}$, cf. relation (3.16) and Corollary 3.3, there holds

$$\underline{\omega}_{0,\iota}(\underline{\alpha}_{\Lambda,x}(\underline{A})) = \lim_\kappa \underline{\omega}_{\lambda_\kappa}(\underline{\alpha}_{\Lambda,x}(\underline{A})) = \lim_\kappa \underline{\omega}_{\lambda_\kappa}(\underline{A}) = \underline{\omega}_{0,\iota}(\underline{A})$$

for all $\underline{A} \in \underline{\mathfrak{A}}$ and $(\Lambda, x) \in \mathcal{P}_+^\uparrow$. The required continuity properties of the functions $(\Lambda, x) \to \underline{\omega}_{0,\iota}(\underline{B}\,\underline{\alpha}_{\Lambda,x}(\underline{A}))$ follow from the norm continuity of the operators $\underline{A} \in \underline{\mathfrak{A}}$ with respect to Poincaré transformations, cf. relation (3.10). To verify that $\underline{\omega}_{0,\iota}$ induces a representation of $\underline{\mathfrak{A}}$ which fulfills the relativistic spectrum condition we have to show that

$$\int dx\, f(x)\, \underline{\omega}_{0,\iota}(\underline{B}\,\underline{\alpha}_x(\underline{A})) = 0 \tag{4.4}$$

for all $\underline{A}, \underline{B} \in \underline{\mathfrak{A}}$ and all functions $f \in L^1(\mathbb{R}^4)$ whose Fourier transforms have support in $\mathbb{R}^4 \backslash \overline{V}_+$. In view of the continuity of the action of $\underline{\alpha}_{\mathcal{P}_+^\uparrow}$ and the fact that $\underline{\alpha}_f(\underline{A}) \in \underline{\mathfrak{A}}$ one gets

$$\int dx\, f(x)\, \underline{\omega}_{0,\iota}(\underline{B}\,\underline{\alpha}_x(\underline{A})) = \underline{\omega}_{0,\iota}(\underline{B}\,\underline{\alpha}_f(\underline{A})) = \lim_\kappa \underline{\omega}_{\lambda_\kappa}(\underline{B}\,\underline{\alpha}_f(\underline{A}))\,.$$

Making again use of the fact that each $\underline{\omega}_\lambda$ is a vacuum state on $\underline{\mathfrak{A}}$, it follows that the latter expression is equal to 0 if $\operatorname{supp} \widetilde{f} \subset \mathbb{R}^4 \backslash \overline{V}_+$. This proves relation (4.4).

It remains to be shown that $\underline{\omega}_{0,\iota}$ is a pure state on $\underline{\mathfrak{A}}$, i.e., that it satisfies the clustering condition. This we do as follows. Let $\mathcal{O}_r$ be the double cone with base radius $r > 0$, centered at the origin, and let $\underline{A}, \underline{B} \in \underline{\mathfrak{A}}(\mathcal{O}_r)$ be such that the functions $t \to \underline{\alpha}_t(\underline{A})$ and $t \to \underline{\alpha}_t(\underline{B})$ (where $\underline{\alpha}_t$, $t \in \mathbb{R}$, denotes the translations in time direction) are differentiable in the norm topology. The set of all such operators (for arbitrary $r > 0$) is norm dense in $\underline{\mathfrak{A}}$. Now since each $\underline{\omega}_\lambda$ is a *pure* vacuum state on $\underline{\mathfrak{A}}$ it follows from [16, Eq. 3.4] that for spatial translations $\mathbf{x}$

with $|\mathbf{x}| > 3r$ there holds

$$
\begin{aligned}
&| \, \underline{\omega}_{\lambda_\kappa}(\underline{A}\, \underline{\alpha}_{\mathbf{x}}(\underline{B})) - \underline{\omega}_{\lambda_\kappa}(\underline{A}) \cdot \underline{\omega}_{\lambda_\kappa}(\underline{B}) \, | \\
&\quad \leq \; c \frac{r^3}{|\mathbf{x}|^2} \cdot \Big(\underline{\omega}_{\lambda_\kappa}(\underline{A}\, \underline{A}^*)^{1/2} \, \underline{\omega}_{\lambda_\kappa}(\dot{\underline{B}}^* \dot{\underline{B}})^{1/2} + \underline{\omega}_{\lambda_\kappa}(\underline{B}\, \underline{B}^*)^{1/2} \, \underline{\omega}_{\lambda_\kappa}(\dot{\underline{A}}^* \dot{\underline{A}})^{1/2} \Big) \\
&\quad \leq \; c \frac{r^3}{|\mathbf{x}|^2} \cdot \Big(|| \, \underline{A} \, || \, || \, \dot{\underline{B}} \, || + || \, \dot{\underline{A}} \, || \, || \, \underline{B} \, || \Big) \, ,
\end{aligned}
$$

where c is a universal (model independent) numerical constant and the dot denotes time derivative. (Note that the relevant inequality in [16] contains a misprint.) The upper bound in this estimate vanishes as $|\mathbf{x}|^{-2}$ for $|\mathbf{x}| \rightarrow \infty$, uniformly in κ. This shows that the limit state $\underline{\omega}_{0,\iota}$ is clustering for all pairs of operators $\underline{A}, \underline{B}$ described above. Since these operators are norm dense in $\underline{\mathfrak{A}}$ it follows that $\underline{\omega}_{0,\iota}$ is clustering and hence pure. $\square$

It is an immediate consequence of this lemma that the kernels of the GNS-representations $\underline{\pi}_{0,\iota}$ of $\underline{\mathfrak{A}}$ induced by the states $\underline{\omega}_{0,\iota} \in SL(\underline{\omega})$ are invariant under Poincaré transformations. Hence, as discussed in Sec. 3, there is an automorphic action $\alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow}$ of the Poincaré group on each local net $\mathfrak{A}_{0,\iota}$, given by equation (3.20). Moreover, it follows from the lemma that the projected states $\omega_{0,\iota}$ are pure vacuum states on $\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow}$. Thus all physical states look like vacuum states in the scaling limit.

Definition: The nets $\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow}$ are called *scaling limit nets* and the corresponding states $\omega_{0,\iota}$ *scaling limit states*, $\iota \in \mathbb{I}$.

We summarize the results of this discussion in the following proposition.

Proposition 4.4 *For $\iota \in \mathbb{I}$, let $\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow}$ and $\omega_{0,\iota}$ be the scaling limit nets and scaling limit states derived from some physical state ω on the underlying net $\mathfrak{A}$. Each $\mathfrak{A}_{0,\iota}$ is a local net which is covariant with respect to the Poincaré transformations $\alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow}$ and each $\omega_{0,\iota}$ is a pure vacuum state on $\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow}$. The collection of scaling limit nets and states does not depend on the choice of the underlying physical state ω.*

Let us turn now to the discussion of the ambiguities involved in the definition of the scaling limit of a theory. As was explained, these ambiguities have their origin in the fact that the nets of scaled states $(\underline{\omega}_\lambda)_{\lambda>0}$ do not converge since the scaling algebra $\underline{\mathfrak{A}}$ subsumes the orbits of local operators under the action of an abundance of renormalization group transformations. Loosely speaking, every scaling limit net $\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow}$ and state $\omega_{0,\iota}$ may be attributed to some particular choice of such a transformation. Since the choice of renormalization group transformations should not affect the physical interpretation of the theory at small scales one may however expect that all scaling limit theories describe the *same physics* at small scales in generic cases. Bearing in mind that different, but physically equivalent nets can be identified with the help of net isomorphisms (cf. Sec. 3) we are led to

the following definition.

Definition: The underlying theory is said to have a *unique scaling limit* if all scaling limit nets $(\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow})$, $\iota \in \mathbb{I}$, derived from physical states on $\mathfrak{A}$ are isomorphic. If there exist net isomorphisms which connect also the respective vacuum states $\omega_{0,\iota}$, the theory is said to have a *unique vacuum structure* in the scaling limit.

Making use of this terminology we see that, in view of Proposition 4.4, there are the following principal possibilities for the structure of the scaling limit of a theory.

Classification: Let $(\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow})$, $\iota \in \mathbb{I}$, be the scaling limit nets arising from physical states in the underlying theory. There are the following alternatives:
i) All (global) algebras $\mathfrak{A}_{0,\iota}$, $\iota \in \mathbb{I}$, consist of multiples of the identity. The theory is then said to have a *classical scaling limit*.
ii) All nets $(\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow})$, $\iota \in \mathbb{I}$, are isomorphic and non-abelian. The theory then has a *unique quantum scaling limit*.
iii) Not all of the nets $(\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow})$, $\iota \in \mathbb{I}$, are isomorphic. The theory is then said to have a *degenerate scaling limit*.

Let us comment on the physical significance of these various possibilities. We refer to the first case as "classical" since it corresponds to the situation where the observables $\underline{A}_\lambda$ at small scales λ attain sharp (i.e., non-fluctuating) values in all physical states, as is the case for classical observables in pure states. It is noteworthy that the "classical ideal" in $\underline{\mathfrak{A}}$ (generated by all commutators) is a proper ideal in such theories. We would like to point out, however, that this classical behaviour of the scaling limit with respect to renormalization group transformations may have its origin in an exceptional quantum behaviour of the observables in small regions: due to the uncertainty principle, the energy-momentum transfer of observables in $\mathfrak{A}(\lambda\mathcal{O})$ has to be at least of order λ^{-1} at small scales. But it could be substantially larger and behave for all observables in $\mathfrak{A}(\lambda\mathcal{O})$ like λ^{-q}, say, for some $q > 1$. Then it would not be possible to exhibit sequences of operators $\underline{A}_\lambda$ which occupy a limited volume of phase space at small scales, apart from multiples of the identity (cf. the remark after Lemma 2.2). Hence the scaling limit would be "classical" in such theories. Examples where Planck's constant in the uncertainty principle is effectively "running" in this way at small scales could be non-renormalizable theories.

The second case corresponds to theories which have (in the language of the renormalization group) an ultraviolet fixed point. The simplest example of this kind is free field theory. The corresponding nets can be shown to have a unique scaling limit which, as expected, is the net generated by a massless free field [17]. This example, as trivial as it may be, illustrates the fact that the apparent ambiguities in our construction of the scaling limit disappear if one identifies isomorphic nets. Of particular interest in this second class of models are the asymptotically free theories. In such theories the underlying quantum fields have, according to the folklore, almost canonical dimensions (modified by logarithmic corrections) and become massless free fields in the scaling limit. We therefore propose to characterize such theories in the present algebraic setting by the condition that the "Huygens ideal" in $\underline{\mathfrak{A}}$ (generated by the commutators of all pairs of operators which are localized in strictly timelike separated regions) is annihilated in this limit. Another class of theories with a unique quantum scaling limit will be considered in the subsequent section.

The last possibility according to our classification appears if the structure of the underlying theory at small scales cannot be described in terms of a single theory since it varies continuously if one approaches $\lambda = 0$. One then arrives

by our construction at a collection of non-isomorphic scaling limit theories which describe these various structures. It is of interest that even in this complicated situation the physical states can be interpreted at small scales as vacuum states. But the properties of these vacua depend on the scale.

In the remainder of this section we focus attention on the physically significant class of theories with a unique (quantum) scaling limit. It turns out that, due to this uniqueness, dilations are a geometrical symmetry of the scaling limit nets.

Proposition 4.5 *For $\iota \in \mathbb{I}$, let $\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}^\uparrow_+}$ be the (isomorphic) scaling limit nets in a theory with unique scaling limit. There exist automorphisms $\delta^{(0,\iota)}_\mu$, $\mu \in \mathbb{R}^+$, of the respective nets such that*

$$\delta^{(0,\iota)}_\mu(\mathfrak{A}_{0,\iota}(\mathcal{O})) = \mathfrak{A}_{0,\iota}(\mu\mathcal{O})$$

for all bounded regions $\mathcal{O}$ and $\mu > 0$, and

$$\delta^{(0,\iota)}_\mu \circ \alpha^{(0,\iota)}_{\Lambda,x} = \alpha^{(0,\iota)}_{\Lambda,\mu x} \circ \delta^{(0,\iota)}_\mu$$

for all $\mu > 0$ and $(\Lambda, x) \in \mathcal{P}^\uparrow_+$. Moreover, if the underlying theory has a unique vacuum structure in the scaling limit there holds $\omega_{0,\iota} \circ \delta^{(0,\iota)}_\mu = \omega_{0,\iota}$ for all $\mu > 0$.

Proof: Since for any physical state ω in the underlying theory and any $\mu > 0$ the corresponding sets $(\underline{\omega}_\lambda)_{\lambda>0}$ and $(\underline{\omega}_{\mu\lambda})_{\lambda>0}$ of lifted and scaled states on $\underline{\mathfrak{A}}$ coincide, they have the same limit points. Hence the set $SL(\underline{\omega})$ of scaling limit states is invariant under the (adjoint) action of the scaling transformations $\underline{\sigma}_\mu$. The statement then follows from the assumption that all states in $SL(\underline{\omega})$ induce isomorphic nets and that there exist net isomorphisms connecting the scaling limit states if the theory has a unique vacuum structure in the scaling limit.

To see this, let $\underline{\omega}_{0,\iota} \in SL(\underline{\omega})$ and consider the corresponding net $\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}^\uparrow_+}$ which is (by construction) canonically isomorphic to $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}), \mathrm{Ad}\,\underline{U}_{0,\iota}(\mathcal{P}^\uparrow_+)$. Here $(\underline{\pi}_{0,\iota}, \underline{\mathcal{H}}_{0,\iota}, \underline{\Omega}_{0,\iota})$ is the GNS-representation induced by $\underline{\omega}_{0,\iota}$ and $\underline{U}_{0,\iota}$ the unitary representation of $\mathcal{P}^\uparrow_+$ on $\underline{\mathcal{H}}_{0,\iota}$ (cf. Lemma 4.3). Now $\underline{\omega}_{0,\iota(\mu)} \doteq \underline{\omega}_{0,\iota} \circ \underline{\sigma}_\mu \in SL(\underline{\omega})$ and the corresponding net $\mathfrak{A}_{0,\iota(\mu)}, \alpha^{0,\iota(\mu)}_{\mathcal{P}^\uparrow_+}$ is canonically isomorphic to the net $\underline{\pi}_{0,\iota(\mu)}(\underline{\mathfrak{A}}), \mathrm{Ad}\,\underline{U}_{0,\iota(\mu)}(\mathcal{P}^\uparrow_+)$ on $\underline{\mathcal{H}}_{0,\iota}$, where $\underline{\pi}_{0,\iota(\mu)} = \underline{\pi}_{0,\iota} \circ \underline{\sigma}_\mu$ and $\underline{U}_{0,\iota(\mu)}(\Lambda, x) = \underline{U}_{0,\iota}(\Lambda, \mu x)$ for $(\Lambda, x) \in \mathcal{P}^\uparrow_+$. Moreover, the state $\underline{\omega}_{0,\iota(\mu)}$ is represented in this setting by the vector $\underline{\Omega}_{0,\iota} \in \underline{\mathcal{H}}_{0,\iota}$. Since the nets induced by $\underline{\omega}_{0,\iota}$ and $\underline{\omega}_{0,\iota(\mu)}$ are isomorphic (by the uniqueness of the scaling limit) there exists a net isomorphism $\tau^{(0,\iota)}_\mu$ mapping the net $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}), \mathrm{Ad}\,\underline{U}_{0,\iota}(\mathcal{P}^\uparrow_+)$ onto the net $\underline{\pi}_{0,\iota(\mu)}(\underline{\mathfrak{A}}), \mathrm{Ad}\,\underline{U}_{0,\iota(\mu)}(\mathcal{P}^\uparrow_+)$. More explicitly, there holds

$$\tau^{(0,\iota)}_\mu(\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{O}))) = \underline{\pi}_{0,\iota(\mu)}(\underline{\mathfrak{A}}(\mathcal{O})) = \underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mu\mathcal{O}))$$

for all bounded regions $\mathcal{O}$, and

$$\tau^{(0,\iota)}_\mu \circ \mathrm{Ad}\,\underline{U}_{0,\iota}(\Lambda, x) = \mathrm{Ad}\,\underline{U}_{0,\iota(\mu)} \circ \tau^{(0,\iota)}_\mu = \mathrm{Ad}\,\underline{U}_{0,\iota}(\Lambda, \mu x) \circ \tau^{(0,\iota)}_\mu$$

for all $(\Lambda, x) \in \mathcal{P}_+^\uparrow$ and $\mu > 0$. Thus each $\tau_\mu^{(0,\iota)}$ induces an automorphism of the net $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}), \mathrm{Ad}\, \underline{U}_{0,\iota}(\mathcal{P}_+^\uparrow)$ which has the geometrical interpretation of dilations. Moreover, if the theory has a unique vacuum structure in the scaling limit there holds $(\underline{\Omega}_{0,\iota}, \tau_\mu^{(0,\iota)}(\,.\,)\underline{\Omega}_{0,\iota}) = (\underline{\Omega}_{0,\iota}, \,.\, \underline{\Omega}_{0,\iota})$ for some such $\tau_\mu^{(0,\iota)}$. Pulling back $\tau_\mu^{(0,\iota)}$ to the net $\mathfrak{A}_{0,\iota}, \alpha_{\mathcal{P}_+^\uparrow}^{(0,\iota)}$ with the help of the canonical isomorphism mentioned above one obtains automorphisms $\delta_\mu^{(0,\iota)}$ with the desired properties. $\Box$

Remark: The automorphisms $\delta_\mu^{(0,\iota)}$, $\mu \in \mathbb{R}^+$, in this statement do not necessarily provide a representation of the multiplicative group $\mathbb{R}^+$. But, as is easily checked, there holds

$$\delta_\lambda^{(0,\iota)} \circ \delta_\mu^{(0,\iota)} = \delta_{\lambda\mu}^{(0,\iota)} \circ \zeta^{(0,\iota)}(\lambda, \mu)\,, \quad \lambda, \mu \in \mathbb{R}^+,$$

where $\zeta^{(0,\iota)}(\,.\,,\,.\,)$ is a 2-cocycle on $\mathbb{R}^+$ with values in the group of internal symmetries of the net $\mathfrak{A}_{0,\iota}, \alpha_{\mathcal{P}_+^\uparrow}^{(0,\iota)}$. It is thus a problem of cohomology whether one can find a true representation of the dilations by automorphisms of the scaling limit theory. However, we do not enter into this question here.

5 Scale invariant theories

In the conventional field theoretic setting, dilation invariant theories are invariant (fixed points) under renormalization group transformations. In this section we show that these theories are also invariant under the corresponding transformations in the present setting. This result displays the consistency of our method with the standard approach.

Let us begin by listing the additional assumptions on the underlying theory made in this section.

Dilation invariance: On the underlying net $\mathfrak{A}, \alpha_{\mathcal{P}_+^\uparrow}$ acts a family of automorphisms δ_μ, $\mu \in \mathbb{R}^+$, such that

$$\delta_\mu(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mu\mathcal{O})\,, \quad \mu > 0 \tag{5.1}$$

for all bounded regions $\mathcal{O}$ and

$$\delta_\mu \circ \alpha_{\Lambda,x} = \alpha_{\Lambda,\mu x} \circ \delta_\mu\,, \quad \mu > 0 \tag{5.2}$$

for all $(\Lambda, x) \in \mathcal{P}_+^\uparrow$. Moreover, the underlying vacuum state $\omega(\,.\,) = (\Omega, \,.\, \Omega)$ on $\mathfrak{A}$ is dilation invariant,

$$\omega \circ \delta_\mu = \omega\,, \quad \mu > 0\,. \tag{5.3}$$

Note that these assumptions coincide with the results which we have established in Proposition 4.5 for theories with a unique vacuum structure in the scaling limit. In our subsequent analysis of the scaling limit of dilation invariant theories we make in addition the following assumption.

Haag-Swieca compactness condition: Let $E(\,.\,)$ denote the projections of the spectral resolution of $U\,|\,\mathbb{R}^4$, let Δ be any compact subset of $\mathbb{R}^4$ and let $\mathcal{O}$ be any bounded region. Then the linear map from $\mathfrak{A}(\mathcal{O})$ into $\mathcal{H}$ given by

$$A \to E(\Delta)A\Omega\,, \quad A \in \mathfrak{A}(\mathcal{O}) \tag{5.4}$$

is a compact map between the Banach spaces $\mathfrak{A}(\mathcal{O})$ and $\mathcal{H}$. (For a review of compactness and nuclearity conditions, which impose certain physically motivated constraints on the phase space properties of a theory, cf. [18] and references quoted there.)

Let us now discuss the implications of the existence of the dilations $\delta_{\mathbb{R}^+}$ in the underlying theory. Given any vector state in the underlying vacuum representation, we consider its scaled canonical lift on $\underline{\mathfrak{A}}$ and the resulting net $\underline{\mathfrak{A}}^{\underline{\pi}_\lambda}$, $\alpha^{\underline{\pi}_\lambda}_{\mathcal{P}^\uparrow_+}$. As has been shown in Proposition 3.4, this net is isomorphic to the underlying net $\mathfrak{A}_\lambda$, $\alpha^{(\lambda)}_{\mathcal{P}^\uparrow_+}$ at scale λ. Now the existence of dilations implies that the latter nets are isomorphic for different values of λ, say λ_1 and λ_2, a net isomorphism being given by $\delta_{\lambda_2\lambda_1^{-1}}$, cf. relations (5.1) and (5.2). Hence the nets $\underline{\mathfrak{A}}^{\underline{\pi}_\lambda}$, $\alpha^{\underline{\pi}_\lambda}_{\mathcal{P}^\uparrow_+}$, obtained from the original state by renormalization group transformations, describe the same physics at each scale, i.e., the theory is invariant (a "fixed point") under the action of the renormalization group.

It is not quite as simple to show that this invariance under the renormalization group persists also in the scaling limit. As a matter of fact, we have been able to establish this result only under the additional mild phase-space constraints imposed by the Haag-Swieca compactness condition. We remark that the existence of dilation invariant theories satisfying this condition has been established in [19].

Proposition 5.1 *Suppose that the underlying net $\mathfrak{A}$, $\alpha_{\mathcal{P}^\uparrow_+}$ is dilation invariant and satisfies the Haag-Swieca compactness condition. Then this net has a unique scaling limit and vacuum structure in this limit. In fact, the scaling limit nets $\mathfrak{A}_{0,\iota}$, $\alpha^{(0,\iota)}_{\mathcal{P}^\uparrow_+}$, $\iota \in \mathbb{I}$, are all isomorphic to the underlying net $\mathfrak{A}, \alpha_{\mathcal{P}^\uparrow_+}$ and the scaling limit states $\omega_{0,\iota}$ are mapped onto the underlying vacuum state ω by the respective net isomorphisms.*

Proof: Let $\underline{\omega}$ be the canonical lift on $\underline{\mathfrak{A}}$ of the underlying vacuum state ω on $\mathfrak{A}$, let $\underline{\omega}_{0,\iota} \in SL(\underline{\omega})$ and let $\mathfrak{A}_{0,\iota}$, $\alpha^{(0,\iota)}_{\mathcal{P}^\uparrow_+}$ be the corresponding scaling limit net. As in the proof of Proposition 4.5 it is convenient to make use of the fact that this net is canonically isomorphic to $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}})$, $\mathrm{Ad}\,\underline{U}_{0,\iota}(\mathcal{P}^\uparrow_+)$, where $(\underline{\pi}_{0,\iota}, \underline{\mathcal{H}}_{0,\iota}, \underline{\Omega}_{0,\iota})$ is the GNS-representation of $\underline{\mathfrak{A}}$ induced by $\underline{\omega}_{0,\iota}$.

Let $(\underline{\omega}_{\lambda_\kappa})_{\kappa\in\mathbb{K}}$ be a subnet of $(\underline{\omega}_\lambda)_{\lambda>0}$ which converges in the weak-*-topology to $\underline{\omega}_{0,\iota}$. A net isomorphism ϕ mapping $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}), \mathrm{Ad}\,\underline{U}_{0,\iota}(\mathcal{P}^\uparrow_+)$ onto $\mathfrak{A}, \alpha_{\mathcal{P}^\uparrow_+}$ is obtained by setting for $\underline{A} \in \bigcup_{\mathcal{O}} \underline{\mathfrak{A}}(\mathcal{O})$

$$\phi(\underline{\pi}_{0,\iota}(\underline{A})) \doteq w - \lim_\kappa \, \delta^{-1}_{\lambda_\kappa}(\underline{A}_{\lambda_\kappa})\,. \tag{5.5}$$

We thus have to prove the following statements: (a) The right hand side of (5.5) exists and defines (b) a linear map from the algebra $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}})$ into $\mathcal{B}(\mathcal{H})$ which (c) is *-preserving. (d) The map ϕ intertwines the action of the Poincaré transformations $\mathrm{Ad}\,\underline{U}_{0,\iota}(\mathcal{P}_+^\uparrow)$ and $\alpha_{\mathcal{P}_+^\uparrow}$ and there holds for all bounded regions $\mathcal{O}$, $\phi(\underline{\pi}(\underline{\mathfrak{A}}(\mathcal{O}))) = \mathfrak{A}(\mathcal{O})$. Finally (e), ϕ is multiplicative.

(a) Let $\mathcal{O}$ be any bounded region and let $\underline{A} \in \underline{\mathfrak{A}}(\mathcal{O})$. To simplify notation we set $A_\kappa \doteq \delta_{\lambda_\kappa}^{-1}(\underline{A}_{\lambda_\kappa})$ and note that $A_\kappa \in \mathfrak{A}(\mathcal{O})$ and $||\, A_\kappa\, || \leq ||\, \underline{A}\, ||$ for $\kappa \in \mathbb{K}$. Let $B \in \mathfrak{A}(\mathcal{O}_1)$, where $\mathcal{O}_1$ is any other bounded region. We first show that the net of numbers $(\omega(B\, A_\kappa))_{\kappa\in\mathbb{K}}$ converges: it follows from relations (5.1) and (5.2) that the function $\lambda \to \underline{B}_\lambda \doteq \delta_\lambda(B)$ is an element of $\underline{\mathfrak{A}}(\mathcal{O}_1)$. Hence, using the invariance of ω under dilations we obtain

$$\omega(B\, A_\kappa) = \omega \circ \delta_{\lambda_\kappa}(B\, A_\kappa) = \omega(\underline{B}_{\lambda_\kappa}\, \underline{A}_{\lambda_\kappa}) = \underline{\omega}_{\lambda_\kappa}(\underline{B}\, \underline{A})$$

and consequently the limit exists,

$$\lim_\kappa \omega(B\, A_\kappa) = \underline{\omega}_{0,\iota}(\underline{B}\, \underline{A})\,.$$

Since the vacuum vector Ω is cyclic for the algebra $\bigcup_{\mathcal{O}_1} \mathfrak{A}(\mathcal{O}_1)$ and the family of operators A_κ, $\kappa \in \mathbb{K}$, is uniformly bounded we conclude that $w-\lim_\kappa A_\kappa\Omega$ exists. As Ω is separating for the weak closures $\mathfrak{A}(\mathcal{O})^-$ of the local algebras it follows that also $w-\lim_\kappa A_\kappa$ exists and that the limit is an element of $\mathfrak{A}(\mathcal{O})^-$.

(b) For the proof that the map ϕ is well defined and linear it suffices to show that for $\underline{A} \in \underline{\mathfrak{A}}(\mathcal{O})$, $\underline{\pi}_{0,\iota}(\underline{A}) = 0$ implies $w-\lim_\kappa A_\kappa\Omega = 0$ and consequently $w-\lim_\kappa A_\kappa = 0$ (recall that Ω is separating for $\mathfrak{A}(\mathcal{O})^-$). Now, making again use of the fact that ω is invariant under dilations, there holds

$$\begin{aligned} ||\, \underline{\pi}_{0,\iota}(\underline{A})\underline{\Omega}_{0,\iota}\, ||^2 &= \lim_\kappa \underline{\omega}_{\lambda_\kappa}(\underline{A}^*\, \underline{A}) \\ &= \lim_\kappa \omega(\underline{A}^*_{\lambda_\kappa}\, \underline{A}^*_{\lambda_\kappa}) = \lim_\kappa \omega(\delta_{\lambda_\kappa}^{-1}(\underline{A}^*_{\lambda_\kappa})\delta_{\lambda_\kappa}^{-1}(\underline{A}_{\lambda_\kappa})) \\ &= \lim_\kappa ||\, A_\kappa\Omega\, ||^2\,, \end{aligned}$$

which establishes the desired implication.

(c) That ϕ is *-preserving, i.e.,

$$\phi(\underline{\pi}_{0,\iota}(\underline{A})^*) = \phi(\underline{\pi}_{0,\iota}(\underline{A}))^*\,,$$

follows from the fact that $w-\lim_\kappa A_\kappa = A$ implies $w-\lim_\kappa A_\kappa^* = A^*$.

(d) The proof that ϕ intertwines the respective actions of the Poincaré transformations is accomplished by noting that for any $(\Lambda, x) \in \mathcal{P}_+^\uparrow$ and any $\underline{A} \in \bigcup_{\mathcal{O}} \underline{\mathfrak{A}}(\mathcal{O})$ there holds

$$\begin{aligned} &\phi(\underline{U}_{0,\iota}(\Lambda, x)\underline{\pi}_{0,\iota}(\underline{A})\underline{U}_{0,\iota}(\Lambda, x)^{-1}) = \phi(\underline{\pi}_{0,\iota}(\underline{\alpha}_{\Lambda,x}(\underline{A}))) \\ &\quad = w-\lim_\kappa \delta_{\lambda_\kappa}^{-1}((\underline{\alpha}_{\Lambda,x}(\underline{A}))_{\lambda_\kappa}) = w-\lim_\kappa \delta_{\lambda_\kappa}^{-1} \circ \alpha_{\Lambda,\lambda_\kappa x}(\underline{A}_{\lambda_\kappa}) \\ &\quad = w-\lim_\kappa \alpha_{\Lambda,x} \circ \delta_{\lambda_\kappa}^{-1}(\underline{A}_{\lambda_\kappa}) = w-\lim_\kappa U(\Lambda, x)A_\kappa U(\Lambda, x)^{-1} \\ &\quad = U(\Lambda, x)\left(w-\lim_\kappa A_\kappa\right)U(\Lambda, x)^{-1} = \alpha_{\Lambda,x} \circ \phi(\underline{\pi}_{0,\iota}(\underline{A}))\,. \end{aligned}$$

From this chain of equalities it follows in particular that

$$|| \, \alpha_{\Lambda,x} \circ \phi(\underline{\pi}_{0,\iota}(\underline{A})) - \phi(\underline{\pi}_{0,\iota}(\underline{A})) \, || \leq || \, \underline{\alpha}_{\Lambda,x}(\underline{A}) - \underline{A} \, || \, ,$$

hence all elements of $\phi(\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{O})))$ are norm continuous with respect to the action of $\alpha_{\mathcal{P}_+^\uparrow}$. Since we already know from step (a) that $\phi(\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{O}))) \subset \mathfrak{A}(\mathcal{O})^-$ we conclude that $\phi(\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{O}))) \subset \mathfrak{A}(\mathcal{O})$ as $\mathfrak{A}(\mathcal{O})$ is, according to our assumptions in Section 2, the maximal subalgebra of operators in $\mathfrak{A}(\mathcal{O})^-$ which are norm continuous under Poincaré transformations. The opposite inclusion is obtained by noting, as in step (a), that for $B \in \mathfrak{A}(\mathcal{O})$ the corresponding function $\lambda \to \underline{B}_\lambda \doteq \delta_\lambda(B)$ is an element of $\underline{\mathfrak{A}}(\mathcal{O})$ and there holds $\phi(\underline{\pi}(\underline{B})) = \lim_\kappa \, \delta_{\lambda_\kappa}^{-1}(\underline{B}_{\lambda_\kappa}) = B$. Hence $\phi(\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{O}))) = \mathfrak{A}(\mathcal{O})$.

(e) It remains to establish the multiplicativity of the map ϕ, and it is here where the Haag-Swieca compactness condition enters. The crucial step in the argument is the demonstration that the weakly convergent net of operators $A_\kappa \in \mathfrak{A}(\mathcal{O})$, $\kappa \in \mathbb{K}$, considered in step (a), is even convergent in the strong operator topology. Again it suffices to establish the strong convergence of $A_\kappa \Omega$ since Ω is separating for $\mathfrak{A}(\mathcal{O})^-$ and the family A_κ, $\kappa \in \mathbb{K}$, is uniformly bounded. We begin by noting that

$$\begin{aligned} || \, \alpha_x(A_\kappa) - A_\kappa \, || &= || \, \alpha_x \delta_{\lambda_\kappa}^{-1}(\underline{A}_{\lambda_\kappa}) - \delta_{\lambda_\kappa}^{-1}(\underline{A}_{\lambda_\kappa}) \, || \\ &= \, || \, \delta_{\lambda_\kappa}^{-1}(\alpha_{\lambda_\kappa x}(\underline{A}_{\lambda_\kappa}) - \underline{A}_{\lambda_\kappa}) \, || \leq || \, \underline{\alpha}_x(\underline{A}) - \underline{A} \, || \end{aligned}$$

showing that the functions $x \to \alpha_x(A_\kappa)$ are continuous at $x = 0$, uniformly in $\kappa \in \mathbb{K}$. Hence, as in the proof of Lemma 3.1 one can find for given $\varepsilon > 0$ a compact set $\widetilde{\mathcal{O}} \subset \mathbb{R}^4$ and operators $\widetilde{A}_\kappa \in \widetilde{\mathfrak{A}}(\widetilde{\mathcal{O}})$ such that $|| \, A_\kappa - \widetilde{A}_\kappa \, || < \varepsilon$, $\kappa \in \mathbb{K}$. Consequently

$$|| \, A_\kappa \Omega - E(\widetilde{\mathcal{O}}) A_\kappa \Omega \, || = || \, (1 - E(\widetilde{\mathcal{O}}))(A_\kappa - \widetilde{A}_\kappa)\Omega \, || < \varepsilon \, .$$

Hence in order to prove strong convergence of the net $(A_\kappa \Omega)_{\kappa \in \mathbb{K}}$ it suffices to prove strong convergence of $(E(\widetilde{\mathcal{O}}) A_\kappa \Omega)_{\kappa \in \mathbb{K}}$ for all compact regions $\widetilde{\mathcal{O}}$. Now for any such $\widetilde{\mathcal{O}}$ the latter net forms, by the compactness condition, a strongly precompact subnet of $\mathcal{H}$ since $A_\kappa \in \mathfrak{A}(\mathcal{O})$ and $|| \, A_\kappa \, || \leq || \, \underline{A} \, ||$, $\kappa \in \mathbb{K}$. Whence the weakly convergent nets $(E(\widetilde{\mathcal{O}}) A_\kappa \Omega)_{\kappa \in \mathbb{K}}$ converge also strongly and consequently $(A_\kappa \Omega)_{\kappa \in \mathbb{K}}$ converges strongly, as claimed. Now let $\underline{A}, \underline{B} \in \underline{\mathfrak{A}}(\mathcal{O})$, $\mathcal{O}$ bounded, and let $A_\kappa, B_\kappa \in \mathfrak{A}(\mathcal{O})$ be the corresponding nets introduced in (a). Since both nets are uniformly bounded and convergent in the strong operator topology there holds

$$(s - \lim_\kappa \, A_\kappa) \cdot (s - \lim_\kappa \, B_\kappa) \; = \; s - \lim_\kappa \, A_\kappa \, B_\kappa \; = \; s - \lim_\kappa \, \delta_{\lambda_\kappa}^{-1}((\underline{A} \cdot \underline{B})_{\lambda_\kappa}) \, .$$

This shows that

$$\phi(\underline{\pi}_{0,\iota}(\underline{A})) \cdot \phi(\underline{\pi}_{0,\iota}(\underline{B})) = \phi(\underline{\pi}_{0,\iota}(\underline{AB})) = \phi(\underline{\pi}_{0,\iota}(\underline{A}) \cdot \underline{\pi}_{0,\iota}(\underline{B})) \, ,$$

i.e., ϕ is multiplicative. By continuity, ϕ can be extended to a map from $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}})$ to $\mathfrak{A}$, and it thus has all properties required for a net isomorphism.

The remaining statement about the action of ϕ on the vacuum state ω follows from the following relation for all $\underline{A} \in \underline{\mathfrak{A}}$,

$$\omega \circ \phi(\underline{\pi}_{0,\iota}(\underline{A})) = \lim_{\kappa} \omega \circ \delta_{\lambda_\kappa}^{-1}(\underline{A}_{\lambda_\kappa}) = \underline{\omega}_{0,\iota}(\underline{A}) \, ,$$

where we made once again use of the fact that ω is invariant under dilations. This completes the proof of the proposition. $\square$

We emphasize that the latter result does *not* mean that the nets $(\underline{\omega}_\lambda)_{\lambda>0}$ of states on the scaling algebra converge in the scaling limit. There still exists an abundance of different limit points $\underline{\omega}_{0,\iota}$ on $\underline{\mathfrak{A}}$. But the result illustrates our statement that (a) all limit points describe the same physics in generic cases and (b) that the choice of renormalization group transformations does not affect the physical interpretation.

6 Geometric modular action and scaling limit

The scaling limit nets and states provide information about the properties of physical states at very small scales. Of particular interest are the particle structure and the symmetries appearing in this limit. In this section we present a structural result which provides a basis for such an analysis. We will show that if the underlying theory satisfies a condition of geometrical modular action invented by Bisognano and Wichmann, then the scaling limit has this property, too. As a consequence, the scaling limit nets comply with the condition of (essential) Haag duality and one can apply the methods of Doplicher and Roberts [20] to determine from these nets the charged fields and the global gauge group appearing in the scaling limit. These applications will be discussed elsewhere.

Another interesting consequence of this result is the insight that, excluding the case of theories with a classical scaling limit, the local von Neumann algebras $\mathfrak{A}(\mathcal{O})^-$ of the underlying nets are of type III_1 for any double cone $\mathcal{O}$. This reproduces a theorem by Fredenhagen [6] in the present general setting.

We begin by explaining some notation. Introducing proper coordinates on $\mathbb{R}^4$, we choose two lightlike vectors $e_\pm \doteq (\pm 1, 1, 0, 0)$ and consider the two opposite wedge-shaped regions $\mathcal{W}_\pm$ (wedges, for short), given by

$$\mathcal{W}_+ = -\mathcal{W}_- \doteq \{x \in \mathbb{R}^4 : x \cdot e_\pm > 0\} \, . \tag{6.1}$$

These regions are mutually spacelike, $(\mathcal{W}_+)' = \mathcal{W}_-$, and invariant under the one parametric group Λ_t, $t \in \mathbb{R}$, of proper Lorentz transformations ("boosts") fixed by $\Lambda_t e_\pm = \mathrm{e}^{\pm t} e_\pm$. We also consider the discrete Lorentz transformation j which satisfies $j e_\pm = -e_\pm$ and acts like the identity on the edge $(e_\pm)^\perp$ of the wedges $\mathcal{W}_\pm$. Thus $j^2 = 1$ and $j\mathcal{W}_+ = \mathcal{W}_-$.

Within the underlying theory we define the C^*-algebras $\mathfrak{A}(\mathcal{W}_\pm)$ as inductive limits of all local algebras $\mathfrak{A}(\mathcal{O})$, where $\mathcal{O}$ is bounded and $\overline{\mathcal{O}} \subset \mathcal{W}_\pm$. The definition of the corresponding scaling algebras $\underline{\mathfrak{A}}(\mathcal{W}_\pm) \subset \underline{\mathfrak{A}}$ and of the algebras $\mathfrak{A}_{0,\iota}(\mathcal{W}_\pm) \subset \mathfrak{A}_{0,\iota}$ in the scaling limit theory is analogous. The following auxiliary lemma, which

we record here for later use, summarizes some elementary properties of the scaling limit states $\underline{\omega}_{0,\iota}$. Throughout this section we assume that these states are obtained from physical states in the underlying theory.

Lemma 6.1 *Let $\underline{\omega}_{0,\iota}$ be any scaling limit state on $\underline{\mathfrak{A}}$ and let $(\underline{\pi}_{0,\iota}, \underline{\mathcal{H}}_{0,\iota}, \underline{\Omega}_{0,\iota})$ be the corresponding GNS-representation.*
i) The vector $\underline{\Omega}_{0,\iota}$ is cyclic and separating for $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_\pm))^-$.
ii) If the representation is non-trivial, i.e., if the dimension of $\underline{\mathcal{H}}_{0,\iota}$ is greater than one, then $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_\pm))^-$ is a factor of type III_1.

Proof: i) It follows from the relativistic spectrum condition (cf. Lemma 4.3) and a well-known argument due to Reeh and Schlieder [21] that

$$\overline{\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_\pm))\underline{\Omega}_{0,\iota}} = \overline{\bigcup_x \underline{\pi}_{0,\iota}(\underline{\alpha}_x(\underline{\mathfrak{A}}(\mathcal{W}_\pm)))\underline{\Omega}_{0,\iota}}\,.$$

The set on the right hand side contains the vector states in $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{O}))\underline{\Omega}_{0,\iota}$ for all bounded regions $\mathcal{O}$. Since $\underline{\Omega}_{0,\iota}$ is cyclic for $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}})$ by the GNS-construction, the cyclicity of $\underline{\Omega}_{0,\iota}$ for $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_\pm))^-$ follows. That $\underline{\Omega}_{0,\iota}$ is also separating for these algebras is then a consequence of locality.
ii) This statement is also based on the relativistic spectrum condition which implies that the centralizer of the algebras $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_\pm))^-$ relative to the vacuum state consists of multiples of the identity. Cf. references [22, 23] for a proof. $\square$

In the following we assume that the modular operators (introduced in the Tomita-Takesaki theory [24]) which are affiliated with the vacuum state Ω and the algebra $\mathfrak{A}(\mathcal{W}_+)^-$ of the underlying theory act like Lorentz transformations on the underlying net.

Condition of geometric modular action: Let Δ, J be the modular operator and modular conjugation associated with the pair $\mathfrak{A}(\mathcal{W}_+)^-, \Omega$ in the underlying theory. For each bounded region $\mathcal{O} \subset \mathbb{R}^4$ there holds

$$J\,\mathfrak{A}(\mathcal{O})^- J = \mathfrak{A}(j\mathcal{O})^-\,. \tag{6.2}$$

Moreover, the modular objects Δ, J are related to the underlying Poincaré transformations $U(\mathcal{P}_+^\uparrow)$ according to

$$J\,U(\Lambda, x)J = U(j\Lambda j, jx)\,, \quad (\Lambda, x) \in \mathcal{P}_+^\uparrow, \tag{6.3}$$

and, with Λ_t as defined above,

$$\Delta^{it} = U(\Lambda_{2\pi t})\,, \quad t \in \mathbb{R}\,. \tag{6.4}$$

This tight relation between modular objects and geometrical symmetries was first established by Bisognano and Wichmann for local nets which arise from an underlying Wightman theory [5]. In two dimensions the condition has also been established in a purely algebraic setting [25], and in physical spacetime it is a consequence of much weaker assumptions [26, 27]. Therefore, the condition may

be viewed as being "generic" in quantum field theory. It is linked to the existence of a PCT-operator Θ which is related to the modular conjugation J in the above condition according to

$$J = U(R)\Theta\,, \tag{6.5}$$

where R denotes the rotation about the 1-axis by the angle π.

If the underlying theory has this property there holds, for any $A \in \mathfrak{A}(\mathcal{O})$, $JAJ \in \mathfrak{A}(j\mathcal{O})$ (since $\mathfrak{A}(\mathcal{O})$ is the maximal subalgebra in $\mathfrak{A}(\mathcal{O})^-$ of operators which are norm continuous under Poincaré transformations) and, for any $A \in \mathfrak{A}$, $J\alpha_{\Lambda,x}(A)J = \alpha_{j\Lambda j,jx}(JAJ)$. Using these facts we introduce an anti-automorphism $\underline{\chi} : \underline{\mathfrak{A}} \to \underline{\mathfrak{A}}$ by setting

$$(\underline{\chi}(\underline{A}))_\lambda \doteq J\,\underline{A}_\lambda J\,, \quad \lambda > 0\,. \tag{6.6}$$

Relations (6.2) and (6.3) imply that

$$\underline{\chi}(\underline{\mathfrak{A}}(\mathcal{O})) = \underline{\mathfrak{A}}(j\mathcal{O}) \tag{6.7}$$

for all bounded regions $\mathcal{O}$ and

$$\underline{\chi} \circ \underline{\alpha}_{\Lambda,x} = \underline{\alpha}_{j\Lambda j,jx} \circ \underline{\chi}\,, \quad (\Lambda, x) \in \mathcal{P}_+^\uparrow\,. \tag{6.8}$$

It is also clear from its definition that $\underline{\chi}$ commutes with the scaling transformations,

$$\underline{\chi} \circ \underline{\sigma}_\mu = \underline{\sigma}_\mu \circ \underline{\chi}\,, \quad \mu > 0\,. \tag{6.9}$$

Since J is antiunitary and $J\Omega = \Omega$ there holds $(\Omega, JAJ\Omega) = (\Omega, A^*\Omega)$, $A \in \mathfrak{A}$. Hence, if $\omega(\,.\,) = (\Omega, \,.\,\Omega)$ denotes the vacuum state and $\underline{\omega}$ its canonical lift on $\underline{\mathfrak{A}}$ it follows that

$$\underline{\omega} \circ \underline{\chi}(\underline{A}) = \underline{\omega}(\underline{A}^*) = \overline{\underline{\omega}(\underline{A})}\,, \quad \underline{A} \in \underline{\mathfrak{A}}\,, \tag{6.10}$$

i.e., $\underline{\omega}$ is skew-invariant under the adjoint action of $\underline{\chi}$.

Now let $\underline{\omega}_{0,\iota} \in SL(\underline{\omega})$ be any scaling limit state on $\underline{\mathfrak{A}}$, let $(\underline{\pi}_{0,\iota}, \underline{\mathcal{H}}_{0,\iota}, \underline{\Omega}_{0,\iota})$ be its GNS-representation and let $\underline{U}_{0,\iota}$ be the unitary representation of $\mathcal{P}_+^\uparrow$ on $\underline{\mathcal{H}}_{0,\iota}$, cf. Lemma 4.3. It is our aim to show that the corresponding scaling limit net complies with the principle of geometric modular action. To this end we consider the modular operator and conjugation $\underline{\Delta}_{0,\iota}, \underline{J}_{0,\iota}$ associated with the pair $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_+))^-, \underline{\Omega}_{0,\iota}$. (Note that these operators are well-defined in view of the first part of Lemma 6.1.) In the following crucial lemma we show that they are related to the Poincaré transformations.

Lemma 6.2 *Assume that the underlying theory satisfies the condition of geometric modular action. Then it holds that*

$$\underline{\Delta}_{0,\iota}^{it} = \underline{U}_{0,\iota}(\Lambda_{2\pi t})\,, \quad t \in \mathbb{R}\,,$$

where Λ_t*,* $t \in \mathbb{R}$*, are the proper Lorentz transformations introduced above, and*

$$\underline{J}_{0,\iota}\underline{U}_{0,\iota}(\Lambda, x)\underline{J}_{0,\iota} = \underline{U}_{0,\iota}(j\Lambda j, jx)\,, \quad (\Lambda, x) \in \mathcal{P}_+^\uparrow\,.$$

Moreover, the modular conjugation $\underline{J}_{0,\iota}$ *implements the action of* $\underline{\chi}$ *in the representation* $\underline{\pi}_{0,\iota}$*,*

$$\mathrm{Ad}\,\underline{J}_{0,\iota} \circ \underline{\pi}_{0,\iota} = \underline{\pi}_{0,\iota} \circ \underline{\chi}\,.$$

Proof: We use again the fact that $\underline{\omega}_{0,\iota}$ is the weak-*-limit of some convergent subnet $(\underline{\omega}_{\lambda_\kappa})_{\kappa\in\mathbb{K}}$ of the lifted and scaled vacuum state ω. Let $\underline{A}, \underline{B} \in \underline{\mathfrak{A}}(\mathcal{W}_+)$ and consider the functions

$$t \to \underline{\omega}_{\lambda_\kappa}(\underline{A}\,\underline{\alpha}_{\Lambda_t}(\underline{B})) = (\Omega, \underline{A}_{\lambda_\kappa}\alpha_{\Lambda_t}(\underline{B}_{\lambda_\kappa})\Omega)$$

for $\kappa \in \mathbb{K}$. Since $\underline{A}_\lambda, \underline{B}_\lambda \in \mathfrak{A}(\mathcal{W}_+)$ for all $\lambda > 0$ it follows from relation (6.4) and the fundamental results of the Tomita-Takesaki theory [24] that the restriction of each state $\underline{\omega}_{\lambda_\kappa}$ to the C^*-dynamical system $\underline{\mathfrak{A}}(\mathcal{W}_+), \underline{\alpha}_{\Lambda_\mathbb{R}}$ satisfies the KMS-condition at fixed inverse temperature 2π. As a consequence, the weak-*-limit point $\underline{\omega}_{0,\iota}$ of these functionals has this property, too [24, Thm. 5.3.30]. Hence the unitary group $\underline{U}_{0,\iota}(\Lambda_{2\pi t})$, $t \in \mathbb{R}$, which implements the automorphisms $\underline{\alpha}_{\Lambda_{2\pi t}}$ in the representation $\underline{\pi}_{0,\iota}$ is the modular group corresponding to $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_+))^-, \underline{\Omega}_{0,\iota}$. The first equality in the lemma then follows.

In the next step we prove the third equality. Because of the relations (6.9) and (6.10) there holds $\underline{\omega}_\lambda \circ \underline{\chi} = \overline{\underline{\omega}_\lambda}$, $\lambda > 0$, and consequently the limit state $\underline{\omega}_{0,\iota}$ is also skew-invariant under the action of $\underline{\chi}$. Thus there exists on $\underline{\mathcal{H}}_{0,\iota}$ an antiunitary operator $\underline{I}$ which implements the action of $\underline{\chi}$, i.e., $\underline{I}\,\underline{\pi}_{0,\iota}(\underline{A})\underline{I} = \underline{\pi}_{0,\iota}(\underline{\chi}(\underline{A}))$ for all $\underline{A} \in \underline{\mathfrak{A}}$, and which leaves $\underline{\Omega}_{0,\iota}$ invariant. We will show that $\underline{I} = \underline{J}_{0,\iota}$. To this end we pick any $\underline{A} \in \underline{\mathfrak{A}}$ and any $\underline{B}^* \in \underline{\mathfrak{A}}(\mathcal{W}_+)$ which is entire analytic with respect to the action of $\underline{\alpha}_{\Lambda_\mathbb{R}}$. Since $\mathcal{W}_+$ is invariant under the action of the Lorentz transformations $\Lambda_\mathbb{R}$ it is apparent that the set of all such operators is norm dense in $\underline{\mathfrak{A}}(\mathcal{W}_+)$. Now, as $\underline{\alpha}_{\Lambda_{i\pi}}(\underline{B}^*) \in \underline{\mathfrak{A}}(\mathcal{W}_+)$, we have

$$\begin{aligned}
(\underline{\Omega}_{0,\iota}, \underline{\pi}_{0,\iota}(\underline{A})\underline{U}_{0,\iota}(\Lambda_{i\pi})\underline{\pi}_{0,\iota}(\underline{B}^*)\underline{\Omega}_{0,\iota}) &= (\underline{\Omega}_{0,\iota}, \underline{\pi}_{0,\iota}(\underline{A})\underline{\pi}_{0,\iota}(\underline{\alpha}_{\Lambda_{i\pi}}(\underline{B}^*))\underline{\Omega}_{0,\iota}) \\
&= (\underline{\Omega}_{0,\iota}, \underline{\pi}_{0,\iota}(\underline{A}\,\underline{\alpha}_{\Lambda_{i\pi}}(\underline{B}^*))\underline{\Omega}_{0,\iota}) = \lim_\kappa\,(\Omega, \underline{A}_{\lambda_\kappa}(\underline{\alpha}_{\Lambda_{i\pi}}(\underline{B}^*))_{\lambda_\kappa}\Omega) \\
&= \lim_\kappa\,(\Omega, \underline{A}_{\lambda_\kappa}\alpha_{\Lambda_{i\pi}}(\underline{B}^*_{\lambda_\kappa})\Omega) = \lim_\kappa\,(\Omega, \underline{A}_{\lambda_\kappa}U(\Lambda_{i\pi})\underline{B}^*_{\lambda_\kappa}\Omega) \\
&= \lim_\kappa\,(\Omega, \underline{A}_{\lambda_\kappa}J\underline{B}_{\lambda_\kappa}\Omega) = \lim_\kappa\,(\Omega, \underline{A}_{\lambda_\kappa}(\underline{\chi}(\underline{B}))_{\lambda_\kappa}\Omega) \\
&= (\underline{\Omega}_{0,\iota}, \underline{\pi}_{0,\iota}(\underline{A}\,\underline{\chi}(\underline{B}))\underline{\Omega}_{0,\iota}) = (\underline{\Omega}_{0,\iota}, \underline{\pi}_{0,\iota}(\underline{A})\underline{\pi}_{0,\iota}(\underline{\chi}(\underline{B}))\underline{\Omega}_{0,\iota})\,,
\end{aligned}$$

where in the sixth equality we made use of relation (6.4) and the Tomita-Takesaki theory. Since $\underline{A} \in \underline{\mathfrak{A}}$ was arbitrary, it follows from the above equation and the preceding result that

$$\underline{J}_{0,\iota}\underline{\pi}_{0,\iota}(\underline{B})\underline{\Omega}_{0,\iota} = \underline{U}_{0,\iota}(\Lambda_{i\pi})\underline{\pi}_{0,\iota}(\underline{B}^*)\underline{\Omega}_{0,\iota} = \underline{\pi}_{0,\iota}(\underline{\chi}(\underline{B}))\underline{\Omega}_{0,\iota} = \underline{I}\,\underline{\pi}_{0,\iota}(\underline{B})\underline{\Omega}_{0,\iota}\,.$$

But the set of vectors $\underline{\pi}_{0,\iota}(\underline{B})\underline{\Omega}_{0,\iota}$, where $\underline{B} \in \underline{\mathfrak{A}}(\mathcal{W}_+)$ is analytic, is dense in $\underline{\mathcal{H}}_{0,\iota}$, cf. Lemma 6.1, and since $\underline{J}_{0,\iota}$ and $\underline{I}$ are bounded operators it follows that $\underline{I} = \underline{J}_{0,\iota}$,

as claimed. The second equality in the lemma is now an immediate consequence of relation (6.8). □

It follows from this result and the Tomita-Takesaki theory that the scaling limit nets comply with the condition of geometric modular action. For the precise formulation of this result we have to consider the representations of the nets $\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow}$ which are induced by the corresponding scaling limit states $\omega_{0,\iota}$, $\iota \in \mathbb{I}$. Since these representations are the defining representations of the respective nets we do not indicate in the following statement the pertinent homomorphisms for the sake of a simpler notation.

Proposition 6.3 *Given an underlying theory which complies with the condition of geometric modular action, let* $\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow}$ *be the associated scaling limit nets in the defining representation induced by* $\omega_{0,\iota}$*, and let* $U_{0,\iota}(\mathcal{P}_+^\uparrow)$ *and* $\Omega_{0,\iota}$ *be the respective Poincaré transformations and the GNS-vector corresponding to* $\alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow}$ *and* $\omega_{0,\iota}$*,* $\iota \in \mathbb{I}$*.*
(i) The modular operators $\Delta_{0,\iota}$ *and conjugations* $J_{0,\iota}$ *fixed by the pairs* $\mathfrak{A}_{0,\iota}(\mathcal{W}_+)^-$*,* $\Omega_{0,\iota}$ *satisfy relations (6.2) to (6.4) with reference to the respective nets* $\mathfrak{A}_{0,\iota}$ *and representations* $U_{0,\iota}$ *of* $\mathcal{P}_+^\uparrow$*.*
(ii) The nets $\mathfrak{A}_{0,\iota}$ *satisfy wedge-duality, i.e.*

$$\mathfrak{A}_{0,\iota}(\mathcal{W}_+)' = \mathfrak{A}_{0,\iota}(\mathcal{W}_-)^- \, .$$

Proof: Any net $\mathfrak{A}_{0,\iota}, \alpha^{(0,\iota)}_{\mathcal{P}_+^\uparrow}$ is, by its very definition, isomorphic to the net $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}})$, $\mathrm{Ad}\, \underline{U}_{0,\iota}(\mathcal{P}_+^\uparrow)$ on $\underline{\mathcal{H}}_{0,\iota}$, and the corresponding isomorphism connects the states $\omega_{0,\iota}$ and $(\underline{\Omega}_{0,\iota}, \, . \, \underline{\Omega}_{0,\iota})$. The first part of the statement then follows from the preceding lemma and the geometric action of $\underline{\chi}$ given in relation (6.7). As a consequence,

$$\mathfrak{A}_{0,\iota}(\mathcal{W}_-)^- = J_{0,\iota} \mathfrak{A}_{0,\iota}(\mathcal{W}_+)^- J_{0,\iota} = \mathfrak{A}_{0,\iota}(\mathcal{W}_+)' \, ,$$

where the second equality holds according to the Tomita-Takesaki theory. This proves the second part of the statement. □

This result shows that the property of geometric modular action of the underlying theory persists in the scaling limit. In view of Proposition 4.5 and results by Guido and Longo [27] one may expect that this property can also directly be established in the scaling limit in theories of physical interest, such as asymptotically free theories. Its consequence, wedge duality and, as a further implication, (essential) Haag duality [28], is an important ingredient in the Doplicher-Haag-Roberts approach to the superselection analysis [9]. As another interesting consequence of wedge duality in the scaling limit we establish the following result on the type of local algebras in the underlying theory.

Proposition 6.4 *Suppose that the underlying net* $\mathfrak{A}$ *has a non-classical scaling limit, cf. Sec. 4, which satisfies wedge-duality. Then the local von Neumann algebras* $\mathfrak{A}(\mathcal{O})^-$ *are of type* III_1 *for all double cones* $\mathcal{O}$*.*

Proof: We quote the following criterion for the type III_1-property, originally established by Fredenhagen [6], from [29, Sec. 16.2].

Criterion: Let $\mathcal{M}$ be a von Neumann algebra on some Hilbert space with separating vector[4] Ω. The algebra $\mathcal{M}$ is of type III_1 if, given $\vartheta \in \mathbb{R}_+$ and $\varepsilon > 0$, there exist nets $A_\kappa \in \mathcal{M}$, $B_\kappa \in \mathcal{M}'$, $\kappa \in \mathbb{K}$, such that

$$|| \, A_\kappa \Omega \, || \geq 1 \, , \quad || \, (\vartheta^{1/2} B_\kappa - A_\kappa^*) \Omega \, || < \varepsilon \, , \quad || \, (B_\kappa^* - \vartheta^{1/2} A_\kappa) \Omega \, || < \varepsilon$$

and $A_\kappa^* A_\kappa$, $B_\kappa^* B_\kappa$ and $B_\kappa A_\kappa$ tend weakly to multiples of 1.

We shall exhibit operators A_κ, B_κ with the required properties by making use of the assumption that there exists some scaling limit state $\underline{\omega}_{0,\iota}$ which induces a non-trivial representation $(\underline{\pi}_{0,\iota}, \underline{\mathcal{H}}_{0,\iota}, \underline{\Omega}_{0,\iota})$ of $\underline{\mathfrak{A}}$. By the second part of Lemma 6.1 the algebras $\underline{\pi}_{0,\iota}(\mathcal{W}_\pm)^-$ are then factors of type III_1, and taking into account the condition of wedge duality we get, using the Tomita-Takesaki theory,

$$\underline{J}_{0,\iota} \underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_+))^- \underline{J}_{0,\iota} = \underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_+))' = \underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_-))^- \, ,$$

where $\underline{\Delta}_{0,\iota}, \underline{J}_{0,\iota}$ are the modular operator and conjugation corresponding to the pair $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_+))^-, \underline{\Omega}_{0,\iota}$. We note that, by Kaplansky's density theorem [30], the domain $\underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_+))\underline{\Omega}_{0,\iota}$ is a core for the modular operator $\underline{\Delta}_{0,\iota}^{1/2}$, and by taking into account that $\underline{\mathfrak{A}}(\mathcal{W}_+)$ is generated by its local subalgebras, the same holds true for the domain $\bigcup_{\overline{\mathcal{O}} \subset \mathcal{W}_+} \underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{O}))\underline{\Omega}_{0,\iota}$, where the regions $\overline{\mathcal{O}}$ are compact.

In view of these facts it follows from the very definition of von Neumann algebras of type III_1 that for each $\vartheta \in \mathbb{R}_+$ and $\varepsilon > 0$ there exists an operator $\underline{A} \in \underline{\mathfrak{A}}(\mathcal{O}_1)$, where $\overline{\mathcal{O}_1} \subset \mathcal{W}_+$ is compact, such that

$$|| \, (\underline{\pi}_{0,\iota}(\underline{A})^* - \vartheta^{1/2} \underline{J}_{0,\iota} \underline{\pi}_{0,\iota}(\underline{A}) \underline{J}_{0,\iota}) \underline{\Omega}_{0,\iota} \, || = || \, (\underline{\Delta}_{0,\iota}^{1/2} - \vartheta^{1/2}) \underline{\pi}_{0,\iota}(\underline{A}) \underline{\Omega}_{0,\iota} \, || < \varepsilon$$

while

$$|| \underline{\pi}_{0,\iota}(\underline{A}) \underline{\Omega}_{0,\iota} \, || > 1 \, .$$

By wedge duality one has $\underline{J}_{0,\iota} \underline{\pi}_{0,\iota}(\underline{A}) \underline{J}_{0,\iota} \in \underline{\pi}_{0,\iota}(\underline{\mathfrak{A}}(\mathcal{W}_-))^-$, and applying Kaplansky's theorem a second time we see that there exists an operator $\underline{B} \in \underline{\mathfrak{A}}(\mathcal{O}_2)$, where $\overline{\mathcal{O}}_2 \subset \mathcal{W}_-$ is compact, such that

$$|| \, (\underline{\pi}_{0,\iota}(\underline{B}) - \underline{J}_{0,\iota} \underline{\pi}_{0,\iota}(\underline{A}) \underline{J}_{0,\iota}) \underline{\Omega}_{0,\iota} \, || < \varepsilon \text{ and } || \, (\underline{\pi}_{0,\iota}(\underline{B})^* - \underline{J}_{0,\iota} \underline{\pi}_{0,\iota}(\underline{A})^* \underline{J}_{0,\iota}) \underline{\Omega}_{0,\iota} \, || < \varepsilon .$$

Combining these estimates with the preceding inequalities and making use of the fact that $\underline{J}_{0,\iota}$ is an anti-unitary involution, we arrive at

$$\begin{aligned} || \, (\underline{\pi}_{0,\iota}(\underline{A})^* - \vartheta^{1/2} \underline{\pi}_{0,\iota}(\underline{B})) \underline{\Omega}_{0,\iota} \, || &\leq (1 + \vartheta^{1/2}) \varepsilon \quad \text{and} \\ || \, (\underline{\pi}_{0,\iota}(\underline{B})^* - \vartheta^{1/2} \underline{\pi}_{0,\iota}(\underline{A})) \underline{\Omega}_{0,\iota} \, || &\leq (1 + \vartheta^{1/2}) \varepsilon \, . \end{aligned}$$

Bearing in mind that the GNS-representation $(\underline{\pi}_{0,\iota}, \underline{\mathcal{H}}_{0,\iota}, \underline{\Omega}_{0,\iota})$ is induced by a scaling limit state $\underline{\omega}_{0,\iota}$ which in turn is the weak-*-limit point of a net $(\underline{\omega}_{\lambda_\kappa})_{\kappa \in \mathbb{K}}$

[4] In [29] it was assumed that Ω is also cyclic for $\mathcal{M}$. But this is not necessary since the criterion, as stated here, implies that the stronger version is satisfied in the subspace $\overline{\mathcal{M}\Omega}$.

arising from the underlying vacuum state, we conclude that there is some $\kappa_0' \in \mathbb{K}$ such that for $\kappa > \kappa_0'$ there holds $|| \, \underline{A}_{\lambda_\kappa} \Omega \, || \geq 1$ and

$$||(\underline{A}^*_{\lambda_\kappa} - \vartheta^{1/2} \underline{B}_{\lambda_\kappa})\Omega \, || \leq (1 + \vartheta^{1/2})\varepsilon \, , \quad ||(\underline{B}^*_{\lambda_\kappa} - \vartheta^{1/2} \underline{A}_{\lambda_\kappa})\Omega \, || \leq (1 + \vartheta^{1/2})\varepsilon \, .$$

Since $\underline{A} \in \underline{\mathfrak{A}}(\mathcal{O}_1)$, $\underline{B} \in \underline{\mathfrak{A}}(\mathcal{O}_2)$, we have also $\underline{A}_{\lambda_\kappa} \in \mathfrak{A}(\lambda_\kappa \mathcal{O}_1)$ and $\underline{B}_{\lambda_\kappa} \in \mathfrak{A}(\lambda_\kappa \mathcal{O}_2)$.

Now let $\mathcal{O}$ be any double cone such that $\mathcal{O} \subset \mathcal{W}_+$ and such that the origin 0 lies on its boundary. Since $\overline{\mathcal{O}_1}$ is a compact subset of the (open) wedge $\mathcal{W}_+$, it follows that $\lambda_\kappa \mathcal{O}_1 \subset \mathcal{O}$ for sufficiently small λ_κ, i.e., for all $\kappa > \kappa_0''$ with some suitable $\kappa_0'' \in \mathbb{K}$. Hence $\underline{A}_{\lambda_\kappa} \in \mathfrak{A}(\mathcal{O})^-$ for all of these κ. On the other hand, there holds $\lambda_\kappa \overline{\mathcal{O}_2} \subset \mathcal{W}_-$ for all κ and consequently $\underline{B}_{\lambda_\kappa} \in \mathfrak{A}(\mathcal{W}_-)^- \subset \mathfrak{A}(\mathcal{O})'$, where the inclusion follows from locality. Finally, since $\overline{\mathcal{O}_1}$ and $\overline{\mathcal{O}_2}$ are compact, the weak limits of the nets $\underline{A}^*_{\lambda_\kappa} \underline{A}_{\lambda_\kappa}$, $\underline{B}^*_{\lambda_\kappa} \underline{B}_{\lambda_\kappa}$ and $\underline{A}_{\lambda_\kappa} \underline{B}_{\lambda_\kappa}$ are elements of the algebra $\bigcap_{\mathcal{O} \ni 0} \mathfrak{A}(\mathcal{O})^-$ and therefore multiples of the identity (cf. the proof of Lemma 4.1). Thus, picking $\kappa_0 > \kappa_0', \kappa_0''$ we see that the nets $A_\kappa \doteq \underline{A}_{\lambda_\kappa} \in \mathfrak{A}(\mathcal{O})^-$ and $B_\kappa \doteq \underline{B}_{\lambda_\kappa} \in \mathfrak{A}(\mathcal{O})'$, $\kappa > \kappa_0$, have the properties required by the criterion. Hence $\mathfrak{A}(\mathcal{O})^-$ is of type III_1. But every double cone can be brought by Poincaré transformations into the particular position required in the preceding argument, proving the statement of the proposition. □

7 Outlook

In the present article we have established a general method for the short distance analysis of local nets of observables. Based on the idea that the choice of renormalization group transformations identifying physical observables at different scales ought to be largely arbitrary, apart from a few basic constraints regulating their phase space properties, we were led to the concept of scaling algebra. These scaling algebras are canonically associated with any given net and combine in a convenient way the information about the theory at different scales. It turned out that the scaling algebras have again the structure of local, Poincaré covariant nets on which the renormalization group acts by automorphisms (scaling transformations). These features greatly simplify the short distance analysis since they allow it to apply well-known methods and results from algebraic quantum field theory. Several arguments given in Sections 4 and 6 illustrate this point.

In a forthcoming paper it will be shown how, by using these methods, the phase space properties of the underlying theory which can be expressed in terms of nuclearity and compactness conditions (cf. [18] and references quoted there), determine the phase space structure in the scaling limit. This point is of relevance on one hand for the question of the general nature of the scaling limit ("classical" versus "quantum"). On the other hand it is an important ingredient for its physical interpretation since the phase space properties of a net reveal the causal and thermal features of a theory as well as its particle aspects, cf. [1].

The next step in our programme is the systematic analysis of the superselection and particle structure of the scaling limit theories. Roughly speaking, these theories play a similar role as the asymptotic free field theories in scattering

theory: they provide information about the stable particle content and the symmetries of the theory which become visible in the respective limit; yet they do not contain by themselves any information about the interacting dynamics. This information is gained by *comparing* the properties of the theory at finite scales with those in the scaling limit.

The ultraparticles of a theory, i.e., the particle structures appearing at very small scales such as partons, leptons etc., can be identified with the particles (in the sense of Wigner) of the scaling limit theory [31]. Of course, information about the masses of the ultraparticles gets lost in the scaling limit. But the comparison of the ultraparticle content with the particle content of the underlying theory allows one to decide whether a particle is confined at finite scales, i.e., exists only as an ultraparticle, or disappears in the scaling limit since it disintegrates, say, into non-interacting ultraparticles, as in the case of hadrons. Thus physical ideas and concepts which so far have been studied only in specific models become accessible in our setting to a more general analysis.

Of similar interest is the related problem of the superselection structure of the scaling limit theories. Anticipating that the underlying theory complies with the condition of geometric modular action, given in Sec. 6, we have shown that the scaling limit satisfies the condition of essential Haag duality. Hence, by the fundamental work of Doplicher and Roberts [20] we know that the superselection structure of any scaling limit theory is in one-to-one correspondence to the spectrum (that is, the dual) of a compact group G_0. Moreover, there exist charged Bose- and Fermi fields which transform as tensors under the action of G_0 and generate the charged scaling limit states from the scaling limit vacuum. Again, the symmetry group G and the field content of the underlying theory may be quite different from that of its scaling limit. In asymptotically free theories, cf. Sec. 4, one would expect that the underlying symmetry group G becomes larger in the scaling limit since interactions which can conceal a symmetry at finite scales (confinement) are turned off in the scaling limit. Yet also the opposite phenomenon may occur, e.g., in theories with a classical scaling limit.

The structure of the gauge group G_0 in the scaling limit is of particular interest in asymptotically free gauge theories since it should contain information about the type of the underlying *local* gauge group. Since in the scaling limit there is no difference between global and local gauge transformations, and all charge degrees of freedom should become visible in case of asymptotic freedom, the local gauge group "at a point" should appear as a subgroup of G_0. Thus the scaling algebra is a promising tool for uncovering, from the gauge invariant observables, the full gauge group of a theory.

For the sake of concreteness let us illustrate the above points by the example of the Schwinger model, i.e., massless quantum electrodynamics in two spacetime dimensions, see [32, 33] and references quoted therein. As is well-known, the algebra of observables of this theory is generated (in the defining vacuum representation) by the free massive scalar field ϕ, and all finitely localized observables are elements of the local von Neumann algebras generated by ϕ, or are affiliated with them in the case of unbounded operators. There exists a conserved current

in this model, $j_\mu = \partial^\nu \varepsilon_{\mu\nu}\phi$, but there does not exist any charged superselection sector of states with finite energy, cf. for example [34, Thm. 3.1(1)]. In fact, if one forgets about the origin of the local net of observables in this model, there does not seem to be any particular reason to relate it to a gauge theory.

The picture changes, however, if one proceeds to small scales with the help of the scaling algebra. It turns out that the scaling limit of the theory is a local extension of the net generated by the free scalar massless field (in exponentiated Weyl form). Let us mention as an aside that the use of bounded operators pays off at this point since one does not run into the familiar infrared problems connected with the massless scalar field in two dimensions. Now the interesting point is that the scaling limit theory has a one-parameter family of superselection sectors in the sense of Doplicher, Haag and Roberts which carry a non-trivial charge with respect to the scaling limit of the current j_μ. Thus one encounters in the scaling limit "electrically" charged states which do not appear at finite scales, as is expected in asymptotically free theories exhibiting confinement. It is also worth noting that the vacuum state in the scaling limit is not pure anymore, but the algebra of observables attains a center. This degeneracy of the vacuum is akin to the so-called θ-vacuum structure of the model at finite scales. Thus we recover by our general method from the net of local observables the well-known features of the Schwinger model which are believed to illustrate the structure of physically more interesting theories, such as quantum chromodynamics [35]. A detailed account of these results, as well as a discussion of other simple models, where the scaling limit nets can be computed explicitly, will be published elsewhere.

After the clarification of the possible structure of the scaling limit and its comparison with the underlying theory it would be desirable to understand the more detailed short distance properties of physical states. As we have seen in the present article, for $\lambda \searrow 0$ the leading contribution of the scaled states $\underline{\omega}_\lambda$ on the scaling algebra $\underline{\mathfrak{A}}$ are vacuum states. In next to leading order there ought to appear the ultraparticle content of the given states [31]. Thus by an asymptotic expansion of the functionals $\underline{\omega}_\lambda$ with respect to λ one should be able to establish a general characterization of those physical states which may be viewed as bound states of ultraparticles.

We conclude this outlook with the remark that the scaling algebra provides also a setting for the discussion of the infrared limit $\lambda \to \infty$. Moreover, the general ideas underlying the construction of the scaling algebra should also be useful for the investigation of other scaling limits, such as the non-relativistic limit $c \to \infty$ or the classical limit $\hbar \to 0$. Thus the concept of scaling algebra seems to be a promising tool to tackle a number of interesting physical problems within the setting of algebraic quantum field theory.

Appendix

The ideas of the renormalization group have proven to be useful in the general analysis of quantum field theories on curved spacetimes as well. They are, for example, an important ingredient in the formulation of the principle of local stability, proposed in [36, 37] to chararacterize physical states in those cases where the underlying spacetime structure does not exhibit any symmetries admitting the definition of vacuum states. The renormalization group has also been used to derive the possible values of the Hawking temperature from first principles [36, 38] and to determine the type of local von Neumann algebras [29, 39]. All these interesting results rely on the existence of quantum fields. It is the aim of the present Appendix to indicate how the concept of scaling algebra may serve as a substitute in investigations of the short distance properties of states and observables on arbitrary spacetime manifolds $\mathcal{M}$ with Lorentzian metric g.

Let $(\mathcal{M}, g)$ be a (four-dimensional) spacetime; to simplify the discussion we assume that $(\mathcal{M}, g)$ is globally hyperbolic (cf. [40] for precise definitions). The open, relatively compact subsets (regions) of $\mathcal{M}$ will be denoted by $\mathcal{R}$. Two regions $\mathcal{R}_1, \mathcal{R}_2$ are said to be *spacelike separated* if there does not exist any causal curve with respect to the metric g which connects $\mathcal{R}_1$ and $\mathcal{R}_2$.

A local quantum theory on $(\mathcal{M}, g)$ is fixed by specifying a net $\mathcal{R} \to \mathcal{A}(\mathcal{R})$ of unital C^*-algebras on $\mathcal{M}$ which satisfies the conditions of isotony and locality (spacelike commutativity). We assume that this net is concretely given on some Hilbert-space of "physical states".

It is our aim to analyze the observables and physical states of the underlying theory at small scales. Within the present setting changes of the spatio-temporal scale can conveniently be described in tangent space: let $q \in \mathcal{M}$ be any spacetime point which will be kept fixed in the following and let $T_q\mathcal{M}$ be the tangent space of $\mathcal{M}$ at q. The open subsets with compact closure of $T_q\mathcal{M}$ will be denoted by $\mathcal{O}$. We fix a (sufficiently small, starshaped) neighbourhood $\mathcal{O}_q$ of $0 \in T_q\mathcal{M}$ and map it by the exponential map $\exp_q$ onto a neighbourhood $\mathcal{R}_q$ of $q \in \mathcal{M}$. Similarly, we identify the subsets $\mathcal{O} \subset \mathcal{O}_q$ with the regions $\mathcal{R} = \exp_q(\mathcal{O}) \subset \mathcal{R}_q$. This identification of regions in $\mathcal{M}$ and $T_q\mathcal{M}$ by the use of the exponential map is suggested by the idea that physics at very small scales should be describable in terms of Minkowski space, in accordance with the equivalence principle of general relativity. As is well known, the exponential map complies with this idea in the best possible way: its inverse $\exp_q^{-1}$ induces, from the underlying metric g, a metric on (a suitable neighbourhood of 0 in) $T_q\mathcal{M}$ which coincides with the usual Minkowskian metric at the origin $0 \in T_q\mathcal{M}$, and deviates from it in a neighbourhood of 0 only up to terms of *second* order, in geodesic normal coordinates at q. We also recall that $\exp_q$ maps the straight lines through 0 to the geodesics passing through q.

These geometrical facts suggest that for the purpose of describing the underlying theory at small scales in the neighbourhood $\mathcal{R}_q$ of q it is natural to proceed from the net $\mathcal{A}$ on $\mathcal{M}$ to the associated net $\mathfrak{A}$ on the tangent space $T_q\mathcal{M}$ given

by

$$\mathcal{O} \to \mathfrak{A}(\mathcal{O}) \doteq \mathcal{A}(\exp_q(\mathcal{O} \cap \mathcal{O}_q)), \quad \mathcal{O} \subset T_q\mathcal{M} \tag{A.1}$$

if $\mathcal{O} \cap \mathcal{O}_q \neq \emptyset$. If $\mathcal{O} \cap \mathcal{O}_q = \emptyset$, we put $\mathfrak{A}(\mathcal{O}) = \mathbb{C} \cdot 1$. It is easily checked that the assignment (A.1) is isotonous in $\mathcal{O}$ and hence it defines indeed a net on $T_q\mathcal{M}$ with C^*-inductive limit $\mathfrak{A}$, describing the possible observations in $\mathcal{R}_q$. We emphasize that the net $\mathfrak{A}$ dependes on the choice of the spacetime point $q \in \mathcal{M}$, the neighbourhood $\mathcal{R}_q \subset \mathcal{M}$ (respectively, $\mathcal{O}_q \subset T_q\mathcal{M}$) and, of course, the chosen, physically motivated identification of the regions $\mathcal{O} \subset \mathcal{O}_q$ and $\mathcal{R} \subset \mathcal{R}_q$ via the exponential map. But we refrain from indicating this dependence in the notation.

The geometrical aspects of renormalization group transformations can now be described as in the case of theories on Minkowski space. We consider uniformly bounded functions $\lambda \to \underline{A}_\lambda$, $\lambda > 0$ with values in the algebra of observables $\mathfrak{A}$ and equip them with the algebraic structure and C^*-norm given in relations (3.7) and (3.8). Given any subset $\mathcal{O} \subset T_q\mathcal{M}$, we define the corresponding local algebra by

$$\underline{\mathfrak{A}}(\mathcal{O}) \doteq \{\underline{A} : \underline{A}_\lambda \in \mathfrak{A}(\lambda\mathcal{O}),\, \lambda > 0\}\,. \tag{A.2}$$

The assignment $\mathcal{O} \to \underline{\mathfrak{A}}(\mathcal{O})$ defines a net on $T_q\mathcal{M}$ with corresponding global algebra $\underline{\mathfrak{A}}$ on which scaling transformations $\underline{\sigma}_\mu$, $\mu \in \mathbb{R}^+$, can be defined as in relation (3.14). Hence, lifting the underlying states ω on $\mathfrak{A}$ to the algebra $\underline{\mathfrak{A}}$ by setting

$$\underline{\omega}(\underline{A}) \doteq \omega(\underline{A}_{\lambda=1})\,, \tag{A.3}$$

we can again describe in a convenient way the observations made at small scales by considering the scaled states $\underline{\omega} \circ \underline{\sigma}_\lambda$ for $\lambda \searrow 0$.

The algebra $\underline{\mathfrak{A}}$ contains all functions which comply with the geometrical aspects of renormalization group transformations. But since we did not yet impose any constraints on their phase-space properties, not all of these functions can be interpreted as orbits of local operators under renormalization group transformations (keeping the unit of $\hbar$ fixed). We will indicate below how one can get rid of these unwanted elements by proceeding to suitable subnets of $\underline{\mathfrak{A}}$.

Before entering into that discussion let us mention that already in the present very general setting one can determine the geometrical and causal properties of the nets appearing in the scaling limit: let ω be any state on $\mathfrak{A}$, let $\underline{\omega}$ be its canonical lift on $\underline{\mathfrak{A}}$ and let $\underline{\omega}_{0,\iota}$ be any limit point of the net $(\underline{\omega} \circ \underline{\sigma}_\lambda)_{\lambda>0}$ for $\lambda \searrow 0$. It turns out that the corresponding net $\mathcal{O} \to \mathfrak{A}_{0,\iota}(\mathcal{O}) \doteq \underline{\mathfrak{A}}(\mathcal{O})/\ker(\underline{\pi}_{0,\iota})$, where $\underline{\pi}_{0,\iota}$ is the GNS-representation induced by $\underline{\omega}_{0,\iota}$, can be interpreted as a local net in Minkowski space if one defines on $T_q\mathcal{M}$ the Minkowski metric with respect to any normal geodesic coordinate system at q. Moreover, the scaling limit nets so obtained are independent of the choice of the neighbourhood $\mathcal{O}_q \subset T_q\mathcal{M}$ entering into the definition of $\mathfrak{A}$. This result shows that our approach is fully consistent with the equivalence principle.

Let us now turn to the problem of selecting elements of $\underline{\mathfrak{A}}$ which can be regarded as orbits under the action of the renormalization group. Again it may seem natural to characterize the functions by the requirement that they occupy at each scale λ a fixed volume of "phase space". But the implementation of this idea is not quite as simple as in Minkowski space since there need not exist any isometries of $(\mathcal{M}, g)$, i.e., energy-momentum need not be conserved. Thus, selecting elements of $\underline{\mathfrak{A}}$ by the condition that they have at each scale $\lambda > 0$ a specific energy-momentum transfer involves a certain degree of arbitrariness, as energy-momentum depends in general on the choice of a Cauchy-surface.

If one is primarily interested in the properties of the theory in the scaling limit one can circumvent this problem, however. Namely, it suffices then to control the energy-momentum transfer of the operators $\underline{A}_\lambda$ in the limit $\lambda = 0$, where they are localized at the spacetime point q. This "local" point of view is also suggested by classical field theory, where phase space appears as cotangent bundle of the underlying configuration space.

To implement this idea let us assume that the underlying theory admits (locally) a "dynamics". This requires that there exists a smooth foliation of the region $\mathcal{R}_q$ by Cauchy-surfaces $\mathcal{C}(t)$, where $-t_0 < t < t_0$ denotes the time-parameter (the time function $\mathcal{R}_q \to \mathbb{R}$ being appropriately normalized), and $q \in \mathcal{C}(0)$. The dynamics is then given by a family of endomorphisms α_{t_1,t_2}, $-t_0 < t_2 \leq t_1 < t_0$, of the net $\mathcal{A}$. They map the local algebras $\mathcal{A}(\mathcal{R})$ into $\mathcal{A}(\mathcal{R}_q)$ if $\overline{\mathcal{R}} \subset \mathcal{R}_q$ and t_1, t_2 are sufficiently small and satisfy the equation

$$\begin{aligned} \alpha_{t_1,t_2} \circ \alpha_{t_2,t_3} &= \alpha_{t_1,t_3} \\ \alpha_{t,t} &= \mathrm{id}\,. \end{aligned} \tag{A.4}$$

Let us briefly explain the physical significance of the endomorphisms α_{t_1,t_2} (sometimes called propagators). In suitable representations of $\mathcal{A}$ one may think of α_{t_1,t_2} as time ordered exponential (Dyson expansion) of generators $\delta_t(\,.\,) = i\,[H_t,\,.\,]$, where H_t can locally be interpreted as a Hamiltonian on the Cauchy-surface $\mathcal{C}(t)$. The operators H_t in turn may be thought of in generic cases as suitably regularized integrals of a local stress energy tensor, integrated over $\mathcal{C}(t)$. We need not enter here into the subtle mathematical problems appearing in the rigorous construction of these quantities and only mention that local dynamics have been constructed in models for globally hyperbolic spacetimes, cf. [41].

Let us now return to our problem of characterizing elements of $\underline{\mathfrak{A}}$ with sufficiently regular phase space properties at small scales. Picking any dynamics, and recalling that the fixed point q under scaling transformations lies on the Cauchy-surface $\mathcal{C}(0)$, it seems natural to test the energy momentum transfer of the operators $\underline{A}_\lambda$ at small scales with the help of the endomorphisms $\alpha_{t,0}$ for small t. Thus we propose to select the desired elements of $\underline{\mathfrak{A}}$ by the condition

$$\limsup_{\lambda \searrow 0} \,||\, \alpha_{\lambda t,0}(\underline{A}_\lambda) - \underline{A}_\lambda \,|| \to 0 \quad \text{for} \quad t \to 0\,, \tag{A.5}$$

in analogy to condition (3.5). Thinking of the generators of the dynamics in terms of Cauchy-surface integrals of a local stress energy tensor, it is evident that condition (A.5) does not depend on the shape of $\mathcal{C}(0)$ in the spacelike complement of q.

So the arbitrariness in selecting regular elements from $\underline{\mathfrak{A}}$ largely disappears. By imposing condition (A.5) for the dynamics corresponding to all possible choices of a time direction we arrive at a strictly local generalization of condition (3.5).

It is apparent that condition (A.5) determines a subnet of $\underline{\mathfrak{A}}$ which is larger than its analogue introduced in the case of Minkowski space. For one does not impose any restrictions on the energy momentum transfer of the underlying operators at finite scales. But this is irrelevant for the discussion of the scaling limit where the properties of the elements $\underline{A} \in \underline{\mathfrak{A}}$ at finite scales are not tested. To illustrate this fact let us note that if one imposes in the case of Minkowski space theories conditons (3.5) and (3.6) on the elements of the scaling algebra only in the limit of small λ, as in relation (A.5), one arrives at the same scaling limit(s) of the underlying theory as in our present treatment.

We conclude this Appendix with the remark that there exist interesting variants of condition (A.5). For example, one may select "smooth" elements of $\underline{\mathfrak{A}}$ by imposing the condition

$$\limsup_{\lambda \searrow 0} \, || \, \lambda \cdot \delta_{t=0}(\underline{A}_\lambda) \, || < \infty \, , \tag{A.6}$$

where δ_t are the generators of the dynamics (propagators). Quite another approach to testing the phase space properties of $\underline{\mathfrak{A}}$ can be based on the modular groups associated with suitable physical states and local algebras, cf. [42, Sec. 5]. A more detailed exposition of the formalism of scaling algebras on arbitrary spacetime manifolds as well as some applications will be presented elsewhere.

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